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ABSTRACT

The second volume of this proceedings contains the papers presented at the 21st annual meeting of the Mathematics Education Research Group of Australasia. Full papers include: (1) "Teachers Interpreting Algebra: Teachers' Views about the Nature of Algebra" (Brenda Menzel and David Clarke); (2) "Long Odds: Longitudinal Development of Students' Understanding of Odds" (Jonathan Moritz); (3) "An Arithmetic Game with Strategic Components--Playing Procedures of Primary School Children" (Doris Mosel-Gobel); (4) "Ability Grouping: Some Implications for Building Mathematical Understanding" (Judith Mousley); (5) "Ethnographic Research in Mathematics Education: Using Different Types of Visual Data Refined from Videotapes" (Judith Mousley); (6) "A Research-Based Framework for Assessing Early Multiplication and Division Strategies" (Joanne Mulligan); (7) "The Use of Mathematical Function Plotters in Teaching Introductory Calculus" (Bruce Nevill); (8) "Children's Matching of Melodies and Line Graphs" (Steven Nisbet and John Bain); (9) "Identifying Separate and Connected Knowing in Mathematics Education" (Jude Ocean); (10) "Developing the Angle Concept through Investigations" (Kay Owens); (11) "Is There a Need for a Mathematics Intervention Program in Grades 3 and 4?" (Cath Pearn); (12) "Year 3 Students' Place-Value Misconceptions: Another Look At MAB" (Peter Price); (13) "Teaching Mathematically Gifted Students in Primary School" (Ian Putt); (14) "Orchestrating Different Voices in Student Talk about Infinity: Theoretical and Empirical Analyses" (Peter Renshaw and Raymond Brown); (15) "Measuring Attitudes towards Mathematics in Early Childhood and Primary Teacher Education" (Carolyn Roberts, Patricia Cretchley, and Chris Harman); (16) "A Survey of Graphics Calculator Use in Victorian Secondary Schools" (Alla Routitsky and Patrick Tobin); (17) "The Search for Pattern: Student Understanding of the Table of Values Representation for Function" (Julie Ryan and Julian Williams); (18) "Parent Newsletters Supporting Mathematics in the Junior Primary School" (Jan Savell); (19) "Challenging Beliefs about

Mathematics Learning and Teaching Using an Electronic Learning Community" (Sandra Schuck and Gerry Foley); (20) "Mathematics Textbooks: Messages to Students and Teachers" (Mal Shield); (21) "Problem Solving through Problem Posing: The Experience of Two Teacher Education Students" (Beth Southwell); (22) "Teachers Building Number Sense Amid the Challenges of Change: Some Case Studies" (Len Sparrow and Alistair Mcintosh); (23) "The Invisible Wall Project on Problem Solving Processes: Concepts and Methods of Interpretive Work with High Resolution Data" (Martin Stein); (24) "The Incidence of Misconceptions of Decimal Notation amongst Students Grades 5 to 10" (Vicki Steinle and Kaye Stacey); (25) "Kindergarten Students' Progress in the Count Me In Too Project" (Rita Stewart, Bob Wright, and Peter Gould); (26) "Task Context and Applications at the Senior Secondary Level" (Gloria Stillman); (27) "Different Forms of Mathematical Questions for Different Purposes: Comparing Student Responses to Similar Closed and Open-Ended Questions" (Peter Sullivan, Elizabeth Warren, Paul White, and Stephanus Suwarsano); (28) "Student Choice of Computation Methods" (Paul Swan and Jack Bana); (29) "Mathematical Misconceptions--We Have An Effective Method for Reducing Their Incidence but Will the Improvement Persist?" (Philip Swedosh and John Clarke); (30) "Dynamic Scaffolding and Reflective Discourse: Successful Teaching Styles Observed within a Project To Teach Mathematical Thinking Skills" (Howard Tanner and Sonia Jones); (31) "A Computer Environment To Encourage Versatile Understanding of Algebraic Equations" (Mike Thomas and Diane Hall); (32) "Teacher Beliefs about the Learning and Teaching of Mathematics: Some Comparisons" (Danielle Tracey, Bob Perry, and Peter Howard); (33) "Exploring the Impact of CAS in Early Algebra" (David Tynan and Gary Asp); (34) "Computers Are Taking Mathematics into the Next Century: Gender Differences in the Attitudes of Secondary Mathematics Students to the Use of Computers" (Colleen Vale); (35) The Mathematics, Technology and Science Interface: Implementation in the Middle School" (Grady Venville, John Malone, John Wallace, and Leonie Rennie); (36) "Progressing through the van Hiele Levels with Cabri-geometre" (Jill Vincent); (37) "What Is 'Meaning' in Gender Research?" (Margaret Walshaw); (38) "Students' Understanding of the Concept of the Variable" (Elizabeth Warren); (39) "Numeracy Benchmarks for Years 3 and 5: What about Chance and Data?" (Jane Watson); (40) "'This Is a Funny Game--You Can't Say Who's Going To Win!': Three Case Studies of Children's Probabilistic Thinking" (Jenni Way); (41) "Beliefs about the Use of Calculators in an Upper Primary Mathematics Classroom: A Partial Application of the Theory of Planned Behaviour" (Allan White); (42) "Metacognition within Mathematics: A New and Practical Multi-Method Approach" (Jeni Wilson); (43) "An Overview of a Research-Based Framework for Assessing and Teaching Early Number" (Bob Wright); (44) "Language, Mathematics and Social Disadvantage: A Bourdieuian Analysis of Cultural Capital in Mathematics Education" (Robyn Zevenbergen); (45) "Mathematical Saturation within Workplace Contexts" (Robyn Zevenbergen). (ASK)

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**Mathematics Education Research Group
of Australasia Incorporated**

Teaching Mathematics In New Times

Conference Proceedings
Volume 2

**MERGA 21
1998**

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Mathematics Education Research Group
of Australasia Incorporated

Teaching Mathematics In New Times

Volume 2

Clive Kanes
Merrilyn Goos
Elizabeth Warren
(Editors)

MERGA 21
1998

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Preface

The Twenty First Annual Conference of the Mathematics Education Research Group of Australasia (MERGA) was held on the Gold Coast, Australia, 5 - 8 July, 1998. The conference attracted papers from researchers in Australia, New Zealand, the Pacific Islands, the United States of America and Europe. Contributions from a large number of established researchers - many with substantial international reputations - as well as a significant number of beginning researchers were included in the program. Research studies reported at the conference encompassed a variety of mathematics education related topics and made use of a wide range of research methodologies and theoretical perspectives. Published presentations were organised into Keynote Addresses, Published Papers (including two symposiums), and abstracts of Short Communications.

In accepting presentations for the conference, every effort was made to be as inclusive as possible of the research needs and interests of members of the mathematics research community.

Refereeing processes for the conference were overseen by an Editorial Committee membership of which included Clive Kanes (Convener, Griffith University), Merrilyn Goos (University of Queensland) and Elizabeth Warren (Australian Catholic University).

Procedures adopted by this committee to select papers for presentation were as follows.

Refereeing procedure for Published Papers

In this category, each paper was reviewed by at least two referees. If both made the same recommendation, then this decision was upheld. If the decision was split, the paper went to another referee. The majority decision was then upheld. Where referees made their acceptance of a paper conditional on certain revision of the paper, the Editorial Committee used its discretion to ensure that these tasks were appropriately completed. Referees were asked to heed guidelines for published papers consistent with those set out in the call for papers and provide their written comments in a standard format under the following suggested headings:

1. Statement of problem/issues and discussion of its significance
2. Literature review and theoretical framework
3. Methodology and data analysis (where applicable)
4. Discussion
5. Conclusions and implications
6. General comments on paper

As far as was possible the research interests of referees were taken into account when papers were allocated for review.

Refereeing procedures for Short Communications

Abstracts for Short Communications were reviewed by the Editorial Committee. The aim of these reviews was to ensure the relevance of the proposed communication for the conference.

The aims of the Editorial Committee in adopting the methods indicated above were to facilitate high quality feedback to authors and achieve best possible assurance that the conference proceedings conformed to MERGA research guidelines It was especially hoped that beginning researchers would benefit from the feedback received.

The Committee wishes to thank those who generously agreed to act as referees for the conference proceedings - without the dedication abundantly evident in many of the reviews both the conference, and the mathematics education research community, would have been the poorer.

All effort has been made to present authors' work as it was intended - apologies are made where this has not been successful.

Lastly, on a personal note, I wish to thank my colleagues, especially Merrilyn Goos and Elizabeth Warren, for the help and assistance they provided in the production of this work.

Clive Kanes
Editorial Committee, Convener
Brisbane, Australia
June, 1998

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Bell, Garry
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Teachers interpreting algebra:
Teachers' views about the nature of algebra

Brenda Menzel and David Clarke
Faculty of Education, University of Melbourne

This paper looks at the different ways teachers interpreted and represented algebra for their students. In this study, teachers are viewed as mediating between the student and the mathematics content. The teachers in this study differed widely in their epistemological and pedagogical beliefs about algebra but tended to teach similar content. The results of this study suggest that teachers' choices are a consequence of particular beliefs about algebra and that these beliefs are attributable to cultural influences rather than academic considerations.

Introduction

The research reported in this paper evolved out of a concern that the choices that teachers make with respect to teaching algebra often do not relate to teachers' epistemological and pedagogical knowledge of algebra. Given the emphasis placed on algebra in the Australian mathematics curriculum (both nationally and at the state level), teachers' knowledge of algebra and the strategies they employ to teach it are fundamental to the realisation of national and state curricular goals in mathematics. The provision of appropriate professional development support for teachers, whether pre-service or in-service, must be based on an understanding of how teachers themselves conceptualise algebra, of their pedagogical knowledge in relation to the teaching of algebra, and of the bases for the instructional decisions they make in interpreting algebra content for their students. In this research, the pedagogical content knowledge related to the teaching of algebra was investigated in significant detail for a small number of teachers in a single rural secondary school.

The conception of the teacher as mediating between the student and the content offers a powerful perspective on teaching practice. According to the writings of Vygotsky (1962, 1978) cognition is a mediated activity. The vehicles by which the representations of cultural meanings are transferred to the learner include such social constructions as linguistic signs and discourse, orthographies, and numeration systems. For Vygotsky the cognitive development of the individual is possible only within the collective practices of the social group.

This study focused on teachers' pedagogical content knowledge with respect to algebra. Pedagogical content knowledge in relation to mathematics has been described and discussed by Shulman (1987) and Kieran (1992). With the work of these researchers in mind, this study sought to examined the terms teachers employed to describe the content, structure and nature of algebra and to describe teachers' pedagogical content knowledge and teachers' personal beliefs about algebra.

Pedagogical content knowledge

Shulman (1987) emphasised on several occasions that the teacher who is best able to successfully orchestrate a range of learning opportunities with a mixed interest and ability group of students is the one who has a deep knowledge of the topic, being able to see its many relationships to other topics and relationships of the elements within the topic.

The content, Shulman (1987) suggests, needs to be considered by teachers from the perspective of, for example, what is most important for students to know about. With respect to algebra this means that teachers need to understand what algebra is and how it is different from other forms of mathematics. Teachers need to know what knowledge needs to precede other knowledge within the domain of algebra. Teachers need to understand what students are able to comprehend, that is, what concepts are conceptually more difficult than others, the nature of these concepts and how they relate to each other. Then teachers also need to have knowledge of different representations by which ideas might be presented to students so that students are sufficiently interested and willing to entertain them.

Algebra in the curriculum

Algebra considered from the perspective of the curriculum can usefully be described in terms of what it might look like and what students might be doing. Kieran has accumulated a substantial body of research and theoretical writing on the nature of algebra as content to be taught and learned. Implicit in the writing of Kieran (1994) is that one of the most important conceptions of algebra for teachers, students and users of algebra is making the link between conceptualising the nature of algebra as a way of viewing phenomena (identifying mathematical problems including problems with variables and using the idea of a function) and expressing these in symbolic form (generalising and modelling) and as a process of manipulating these symbolic expressions in order to solve the identified problem. However, writers such as Wheeler (1997) say that algebra described for the curriculum is usually described in terms of algebraic strategies and the concepts that link the strategies to a purpose such as solving equations or graphing a function.

Anticipating possible student difficulties

A key aspect of a teacher's pedagogical content knowledge is an awareness by the teacher of likely student difficulties with particular aspects of algebra and knowledge of teaching strategies by which these might be prevented or overcome.

The difficulties with algebra are often couched in terms of the complexity of thinking involved when utilising algebraic structures and formulating mental models (MacGregor and Stacey, 1995). If we consider algebraic thinking, we need to consider more than just activities such as codifying, manipulating, and proving but also "the interactions and distinctions between formulating the problem equations and manipulating them to find the problems solutions" (Bell 1997, p. 151).

Algebra is like natural language in that there can be different denotations with the same sense and one denotation that can take on more than one meaning. Students will often have difficulty with this flexible interpretative nature of algebra and tend to treat expressions as static pieces of non-mathematical information (Mestre 1988, MacGregor & Stacey 1994c). According to Arzarello, Bazzini and Chiappini (1994, p 1-138), one obstacle to using algebra as a thinking tool concerns the fact that algebra does not use the same "natural" way of thinking as do familiar number problems.

The essential point here is that a significant aspect of a teacher's pedagogical content knowledge resides in the teacher's understanding of the difficulties that students are likely to experience and the strategies by which these might be corrected or prevented. This requires of a teacher that she have knowledge of the content of algebra within the domain of mathematics, of the characteristics of learners and learning as these relate to algebra, and of the representational forms and instructional techniques whereby students' learning of algebra might be promoted. This paper reports a study into the form taken by this pedagogical content knowledge for a small community of secondary mathematics teachers.

Methodology

A study of five mathematics teachers of the classes Years 7, 8, 9 and 10 (1995-1997) was carried out to determine empirically, by observation, questionnaire and interview methods, what each Teacher believed about algebra and teaching algebra. Data was collected over two years to validate information gathered because the researcher recognised that a teacher's expressed beliefs might change from one year to the next in accordance with the unique relationship that develops between the teacher his or her classes and changes in curriculum initiatives such as the *Curriculum and Standards Frameworks* (CSF) (BOS, 1995. 1997).

The following methods were used in this Study:-

- Questionnaire methods both oral and written
- Thinking aloud while engaging in problem solving
- Clinical interviews, both structured and interactive
- Naturalistic observations

A study of five mathematics teachers of the classes Years 7, 8, 9 and 10 (1995-1997) was carried out to determine empirically, by observation, questionnaire and interview methods, what each Teacher believed about algebra and teaching algebra. Data was collected over two years to validate information gathered because the researcher recognised that a teacher's expressed beliefs might change from one year to the next in accordance with the unique relationship that develops between the teacher his or her classes and changes in curriculum initiatives such as the *Curriculum and Standards Frameworks* (CSF) (BOS, 1995. 1997).

It was felt that each of these methods was important in the overall design of the research which took into account what often appeared to be conflicting statements and observations of teachers. Comparisons of the diverse data were then able to be made to enable valid conclusions to be drawn.

Interviews were of two types, "think aloud" and "clinical interview". The "think aloud" method involved the interviewer only taking part to provide a stimulus such as posing a problem and occasionally encouraging the subject to continue with their comments about the nature and importance of algebra. The "think aloud" method required the subject to say everything that comes into his or her head while, for example, solving a challenging problem. The "clinical interview" method involved considerable interaction between the interviewer and researcher. The study design took into account the need to inquire thoroughly into the teachers' thinking about algebra. A fixed set of questions was used as the basis of the interview but additional questions sought to clarify, give examples, provide embellishment or additional information. The goal of the interactive clinical interview was for the interviewer to obtain increasingly more specific information about what the teacher believed and why they made the pedagogical decisions that were observed.

Teachers had great difficulty, when interviewed, in describing such things as preferred mode of working and relating this to teaching practice. However, this particular relationship was central to the study so it was important to support clinical interview data with data from other sources. These included observing teachers working out algebraic problems, listening to the way the teachers explained algebra to students and to the researcher, and reflecting upon the teachers' written solutions to problems and teachers' written and verbal comments about written solutions of problems solved by others. Naturalistic observations were made to observe teachers teaching algebra to ascertain the type of language, illustrations and other mediators teachers used to explain algebra to students. During these observations, field notes were kept along with audio and video recording of teachers' and students' behaviour as it occurred 'spontaneously'. Teachers' behaviours were analysed with respect to the extent that they implied beliefs about algebra and teaching algebra.

Questionnaires were used in the study to collect initial information about what algebra was taught in the Years 7, 8, 9 and 10 classrooms along with what teachers regarded to be algebra content and appropriate ways of solving algebra related problems

Results and Discussion

This study was carried out in a small rural community college (catering for grades P to 12). A consequence of the small size of the school is that all teachers are mixed method teachers, teaching other subjects as well as mathematics. Over the period of the study, all secondary mathematics teachers (other than the researcher - Menzel) were included in the research.

To make the structure of the report as transparent as possible and consistent with the scale of this paper, results have been summarised as bullet points, with some brief elaboration of each point. The implications of each result are discussed briefly as they arise.

- Teachers' remarks about the importance of algebra for school mathematics fell into four broad categories:
    - The students' need for algebra for VCE studies;
    - Algebra as a useful alternative for solving problems;
    - Algebra as an important component in student's cultural education; and
    - Algebra as an important activity for developing thinking skills.

Three teachers explained that being able to do algebra was an important aspect of students' cultural education. They believed that our society expected students to study algebra. In the words of one teacher: "...I regard it [learning algebra] as general knowledge. ... It is probably important to teach all students algebra even though they will always have trouble with the concepts."

- Teachers' descriptions of algebra fell into two broad categories, those that referred to the curriculum and those that made specific reference to the epistemology of algebra.

Teachers' descriptions of the nature and function of algebra illustrated wide variation. Teachers' descriptions of algebra content tended to be in terms of algebra curriculum activities, concepts and skills, procedures and/or teacher action when describing functions, equations and generality but in terms of epistemology when describing the symbolic nature of algebra. In the case of the content of Algebra, the teachers were largely in agreement. Five terms used by all teachers to describe algebra were: Solving equations, finding relationships, symbols, factorising and using rules (such as for factorising quadratics). Possibly this was because the available curriculum documents described material in terms of common topics, and the teachers largely accepted this.

Teachers generally described the content of algebra in terms used to name units of work. For instance one teacher's concept map yielded the following terms: Graphs of quadratics, symbols & abstract information, expanding, simplifying & factorising, straight lines and equations, equations & functions, and spreadsheets, experimenting, understanding, memorising. This corresponded closely to the broad topic headings she used to summarise her algebra course. Her broad description of her Year nine algebra course was:-

symbolising and abstracting
first degree equations, expansion & simplification and
quadratics and line graphs

While actual algebra content conformed to the CSF, the terms by which the teachers chose to express this content revealed some interesting differences in the way they viewed algebra.

Each teacher said that algebraic activity can be identified by its symbolic form but differed in their emphasis on this as a distinguishing characteristic of algebra. The teachers generally believed that algebra is being used when one thinks about how to solve mathematical problems involving variables or unknowns even before any formal writing takes place.

Not all teachers believed that algebraic activities need to involve written symbols. Three of the teachers believed the use of formal symbolic notation is very important for effective algebraic solutions, while the other two teachers believed that effective use can be made of algebraic concepts by utilising students' understanding of algebraic procedures and using personal writing styles.

The use of verbs in both concept maps and interviews about the content of algebra suggested a conception of algebra in terms of skills while the use of nouns suggested a focus on broad concepts. The teachers made frequent use of verbs expanding, simplifying & factorising to describe a specific focus when considering teaching equations and generality. They used nouns to describe functions implying that related algebraic notions were not just carrying out procedures but also included broad concepts such as the notion of variable and understanding relationships. The teachers' descriptions of algebra as an approach to solving problems was usually in reference to the nature of algebra but did not feature highly when they described teaching algebra.

- Teachers differed in the way they viewed Generality.

All Teachers' made some reference to "abstract notations" and "symbols" in their concept maps and some reference to ideas of expressing relationships. One teacher used key words in her concept map of algebra that described Generality as "memorising - expanding, simplifying & factorising techniques ... but also understanding relationships and applying this to real data". The teacher's list specified implicitly her regard for the importance of the expression of empirical data in general terms, and describing relationships. Teachers all made several references to ideas of generalising being an inherent part of what one does when doing algebra.

Differences in the teachers' views about Generality tended to be epistemological rather than based on curriculum content. For instance, the teachers were divided on epistemological grounds in their opinion as to whether verbal description of generalizable mathematical relationships should be regarded as algebraic activity. Teachers did not dispute the importance of verbal activity of this type but only whether such activity might be regarded as algebraic.

Teachers agreed that verbal descriptions should be regarded as an important activity to teach students to write an algebraic expression. However, for one teacher the verbal summation of the problem was seen as preceding the algebraic activity of symbolising as opposed to being an algebraic activity in and of itself while other teachers commented that verbal generalisations about numeric relationships constituted algebraic activity. Generally the teachers regarded generalising as having an integrative role for working with functions or equations.

- Teachers emphasised procedural knowledge for the study of algebraic equations

The idea of "Equations" as an algebraic concept was regarded by all teachers to be of significant importance for the teaching of algebra. The teachers' descriptions of the notion of "equations" centred on procedures such as factorising and backtracking. Teachers focused on curriculum units and emphasised working with formulas and skills such as transposing as important algebraic activity. The teachers agreed that constructing equations was an important aspect of algebra for solving many types of problems. The teachers were divided in their views about the legitimacy of substitution into equations as an algebraic activity. This difference is representative of some of the epistemological differences between teachers. For the purpose of teaching, all teachers endorsed the importance of specific procedural knowledge and solving equations.

- The concept of "Functions" was indicated by all teachers to be important for mathematics, if a lower priority for teaching.

There was a strong suggestion from the teachers that understanding the theory of functions lies in the domain of mathematics while application of functions is in the domain of science and other applied disciplines. This came across strongly when interviewing teachers. Such epistemological distinctions offer key insights into the form in which these teachers compartmentalised algebraic knowledge. Interviews with teachers also indicated that they regarded graphing techniques to be the most important aspect of functions for students to know about. The teachers' statements about the place of functions in the curriculum utilised terms expressing skills, such as sketching graphs from their equations, as opposed to understanding general concepts such as the concept of variable. For instance, one teacher spoke about functions in the following way:

> This [functions] is a simple procedure of taking an equation, making up a table of values and plotting these values. I personally don't see this as a difficult procedure but I guess there may be some students who do struggle with this concept ... Drawing and reading graphs is not difficult"

This teacher's comments emphasised a view he shared with other teachers that his responsibility was in the teaching of procedural rather than relational aspects of functions. Teachers endorsed the importance conceptual knowledge involved in understanding functions but centred on understanding specific skills rather than broad concepts such as "variable" .

- Teachers focused on different concepts and skills

The teachers categorised the algebraic content in two sets of 14 questions differently. The teachers' focus on different aspects of algebra suggested that each teacher might have a different way of approaching a particular mathematical task. For example, the teachers' differed in their classification of the following task:

> Three containers, each identical in size and shape, contain three different quantities of liquid. The first is twice the volume of the second and the same as the third. If the total volume is five litres, how much liquid is in each of the three containers?

The teachers' responses suggested three different views about the essentially algebraic content in the task:

- working with symbolic representations;
- interpreting a situation in symbol form; and,
- operating on algebraic objects.

These differences occurred even though the teachers uniformly suggested that the task essentially involved concepts related to equation-solving, such as 'equality', 'symbolic form' and 'mathematical operations'.

Examination of teachers' responses to the "three container" problem above indicated that two of the teachers' believed symbolising to be a significant or important skill required to solve the problem with one of these two teachers believing that operating with algebraic objects was also important. A third teacher believed that interpreting the problem was the most important skill. This, he explained, was because students will usually experience little difficulty finding a solution having once interpreted the original information correctly. A fourth teacher believed that setting up an equation from an understanding of the relationships inherent in the problem was a significant algebraic activity but that manipulating the equations was the most significant algebraic activity involved in completing the "three container" problem above. The fifth teacher believed that logical reasoning could be used to solve the "three container" problem above and that an algebraic solution would only be evident if an equation was constructed and solved in some formal way.

While all teachers agreed that the "three container" problem above involved primarily the solution of equations, each teacher regarded the problem from a different perspective. Each perspective was related to the way these teachers perceived the problem for the purpose of teaching algebra and therefore was described by teachers in light of the skills they believed the problem might best serve to illustrate.

- Algebra was identified by some teachers as a way of thinking.

Whether a problem involved algebra or not was decided by four teachers partly on the basis of their identification of the "algebraic thinking" they regarded to be involved in the problem. All the teachers commented that algebraic thinking could be described in some form. However, whether algebraic thinking could or should be taught depended upon how it was conceived by the teachers.

One teacher's comment was: "You have logic statements and you have algebraic statements as well ... algebraic thinking is simply using the algebra rules and incorporates a knowledge of the development and application of these rules". A different teacher saw algebraic thinking as general ability to process any mathematically abstract information or ideas. A third teacher restricted her description of algebraic thinking to the "... use [of] algebra for abstract thinking needed to solve many problems in maths". The contrasting views being expressed present algebraic thinking as the understanding required for the manipulation of algebraic symbols or a more general form of abstract mathematical thinking related to identifying fundamental abstractions relevant to a particular problem.

In addition, all the teachers shared a view that one component of algebraic thinking was being able to think logically in order to be able to complete algebraic tasks. Teachers emphasised that the logical thinking needed to have a sound theoretical base that included at least knowing algebraic procedures.

The teachers' descriptions of algebraic thinking differed in emphasis more than in content. The teachers' opinions were rationalised in terms of pedagogy rather than epistemology.

Conclusions

During the study the teachers investigated tended to describe algebra in curricular terms rather than engaging in personal theorising about the nature of algebra. The teachers described algebra concepts and skills in terms of topics such as factorising, solving equations and writing algebraic expressions to represent a mathematical problem or situation. Algebra was described as a mental process involving logical reasoning. This form of logical reasoning was associated by teachers with understanding the logical sense of algebraic procedures and being able to use these procedures. The teachers' conceptions of algebra differed in emphasis from each other.

The teachers' descriptions of algebra included a view that algebra provided an approach to solving problems. Three of the teachers suggested that algebra should not always take a predominant role in the classroom when solving mathematical problems with numerical methods being regarded with equal importance. However, one teacher expressed a view that algebraic methods would often prevail in problem solving because of their efficiency. However, for all teachers algebra provided a structure or framework of concepts from which to approach particular types of mathematical problems.

The teachers did not endorse a dominant role for algebra in the school curriculum to the same extent. Differences among teachers were based on epistemological considerations and concerned opinions about whether numerical methods were more efficient. These differences of opinion related to the teachers' personal preferences with regard to methods of solving problems.

This research has documented the diversity of influences and consequent beliefs contributing to the way in which teachers interpret the nature and purpose of algebra for their students. It is not surprising that teachers' conceptions of algebra should give priority to curricular matters in preference to epistemology. What is noteworthy, however, is the degree of difference in significance that individual teachers attach to particular algebraic concepts and skills. These differences raise the question of how well-equipped teachers of mathematics must be in order to make decisions about instructional priorities. This study suggests that some teachers present algebra as something you do, rather than emphasising the central ideas and concepts that characterise algebra. Teachers adopting the former approach are more likely to give students a limited, utilitarian conception of algebra. Such a conception lacks generality and may seriously restrict students' recognition of connections between this knowledge and both 'real world' situations, in which it might be applied, and with the broad domain of mathematics, in which it should find its meaning.

These findings, albeit of a small group of teachers, suggest that 'pedagogical content knowledge' offers a useful window into the thinking underlying the practices and beliefs of mathematics teachers. Should the findings of this study be typical of conceptions of algebra held by the general mathematics teaching community, the implications for teacher professional development, inservice and preservice training and new curriculum initiatives could be profound. Further analysis of this data, may reveal specific steps by which these concerns about teachers' mediation between student and content might be addressed.

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Long Odds: Longitudinal Development of Student Understanding of Odds

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Student understanding of odds is explored by analysis of response data collected in 1993, 1995 and 1997. Students were asked to interpret a newspaper headline, "North at 7-2". Responses included interpreting the numbers as the score, and in three contexts of expression involving chance, frequency of wins, and betting. Levels of responses were assigned according to the SOLO developmental model. Longitudinal development for individuals was observed, and females tended to interpret the numbers as the score, while males were more likely to respond in the context of chance or betting. Levels of response from Grade 6 and 9 students in 1995 and 1997 were lower than in 1993.

Developing understanding about odds is included in the Australian mathematics curriculum as outlined in *A National Statement on Mathematics for Australian Schools*, under the heading "understand and explain social uses of chance processes" (Australian Education Council, 1991, Statement C1, p. 175). Given the widespread participation of Australians in various forms of gambling, the topic of odds has social relevance. It also offers students a different view of probability that admits a subjective judgement, in contrast to dice experiments involving classical assumptions of equal likelihood of outcomes, or to statistical prediction directly based on frequency data.

A model for the development of understanding of odds was proposed by Moritz, Watson and Collis (1996) using the SOLO theoretical model of Biggs and Collis (1982). Students responded to the item shown in Figure 1. Some students offered no response (NR), while some responses were irrelevant, classified as prestructural (P). Many responses interpreted the numbers as the score in the game, and were classified in the ikonic mode (IK), based on experience in the context of football games. Responses in the concrete symbolic mode were those that responded in one of three contexts of expression, including (1) probability/chance expressions, (2) frequency expressions of number of wins for past games or predicted for future games, and (3) social contexts of betting. For each of these contexts of expression, different SOLO development levels were identified in two cycles according to the sophistication of the expression of measurement, as shown in Table 1. The levels distinguished include expressions of basic uncertainty at the unistructural level (U_1), expressions that qualify likelihood at the multistructural level (M_1), and expressions involving quantification at the relational level (R_1). A second learning cycle (U_2-M_2-R_2) accounted for applications of a ratio concept to chance measurement specifically related to the task of interpreting odds. These levels are closely related to those used by Watson, Collis and Moritz (1997) to describe the development of chance measurement in problem solving settings involving tossing a die and random selection from containers.

North at 7-2

But we can still win match, says coach

What does "7-2" mean in this headline about the North against South football match? Give as much detail as you can.
From the numbers, who would be expected to win the game?

Figure 1. Item related to odds from media survey about Chance and Data.

Table 1
Overview of Response Levels by Three Contexts of Expression (adapted from Moritz, Watson and Collis, 1996)

SOLO Level	Summary description	Context of Expression		
		Chance	Frequency	Betting
P/IK	No expression of uncertainty or prediction; irrelevant response; score			
U_1	Simple uncertainty expressed	7-2 chance Likely	Predicted score	Betting numbers for South
M_1	Uncertainty qualified or attempted quantification	% chance 7 in 2 chance	For every 2 pts, 7 pts (predictive) Past score predicts	Odds: no clear ratio idea
R_1	Uncertainty quantified in ratio measure	3.5 to 1 chance	Won 2 out of 7 games	Votes/rating/ratio Odds: ratio clear
U_2	Consolidated ratio as predictor OR part-part-whole ratio	N has 2/7 chance	If play 7, win 2 Played 9 games won 2 or 7	Bet S2, win S7
M_2	Consolidated ratio as predictor AND part-part-whole ratio	N has 7/9 chance	If play 9, N win 7 N win 7 every 9 games played	Bet S2, win S2 + S7 (North favoured)
R_2	Predictive ratio with correct favoured direction	N has 2/9 chance	If play 9, N win 2	Bet S2, win S7 + S2 (South favoured)

The classification of response levels shown in Table 1 forms the framework for exploring various research questions concerning longitudinal development of student understanding of odds. Are there changes in the SOLO levels or the contexts of expression of responses of individual students assessed longitudinally during the four-year interval 1993-1997, and during the two-year intervals, 1993-1995 and 1995-1997? Are there gender differences in this longitudinal development? Cross-cohort comparisons concern the impact of recent curriculum reform and implementation on student understandings. Have SOLO levels or the contexts of expression of student responses from comparable grades changed over the years 1993, 1995 and 1997? Cross-sectional comparisons address the question of gender differences and change by grade for the pooled data set. The results of Watson and Moritz (1998) addressed these questions for chance measurement in other settings. Responses of individuals generally improved to higher SOLO levels, indicating broad support for the developmental framework. No consistent differences were found between different years at comparable grades. Gender differences favouring males were evident for some secondary school grade levels. As the current study concerns a closely related topic, similar results would be expected. In particular, gender differences favouring males would be expected because odds are commonly used in sports settings which generally seem to have greater following by males.

Method

A media survey (Watson, 1994) using newspaper extracts covering different topics in Chance and Data was originally administered in 1993 to 312 Grade 6 students and 393 Grade 9 students in Tasmanian government schools during 45 minutes of class time. The survey included the item analysed in this study, shown in Figure 1. Only students who attempted this or later items in the survey were included in the analysis. Results were reported by Moritz et al. (1996). In 1995 and 1997, the longitudinal aspect of the study involved surveying the same students where they could be traced in schools two and then four years later. The cross-cohort aspect involved surveying new cohorts of Grade 6 and 9 students in 1995 and 1997. Students in these new cohorts from 1995 were also followed longitudinally in 1997. In some schools, entire classes were surveyed including both new and repeat-survey students, while other schools included only repeat-survey students. Fewer new students participated in 1997 due to increasing time demands on students' school time from other sources.

Responses were analysed using the language analysis software NUD•IST (Qualitative Solutions and Research, 1995). A NUD•IST script (Moritz, 1995) was used to sort all responses for each year, the output of which was checked and manual corrections made in the classification of selected responses. Thus all responses were assigned SOLO levels, and responses in the concrete symbolic mode were also assigned contexts of expression. For responses that included aspects of multiple contexts, the dominant context of the response was used, that is the context of the aspect with the highest SOLO level, or if the levels were equal, then the context most clearly expressed at that level. Analyses involving scoring of responses used a scoring scale of 0 for NR, P and IK, 1 for U_1, 2 for M_1, 3 for R_1, 4 for U_2, 5 for M_2 and 6 for R_2. For longitudinal analysis, the difference in scores between two years for each individual student was used as a measure of longitudinal change. For cross-cohort and cross-sectional comparisons, mean scores were evaluated for all relevant sub-groups.

Results

Responses generally interpreted the headline as a score, or within one of three different contexts of expression: probability expressions, frequency expressions and social context expressions. Responses ranged from irrelevant responses (prestructural for this task), to those at the R_2 level. The results are divided into three sections. The first section explores longitudinal development of responses of individual students repeatedly assessed in different years. Examples of levels of responses are given to explore the nature of the longitudinal development in terms of the development model in Table 1. The second section of results considers cross-cohort comparisons of different students in comparable grades to explore the impact of curriculum change between 1993 and 1997. The third section explores a cross-section of grade levels, and explores gender differences in relation to grade level.

Longitudinal Development

To portray the nature of the developmental change for selected students over time, examples are given to illustrate responses from the same student offered in 1993, 1995, and 1997. The following student's responses illustrate increasingly sophisticated responses, all within the chance context of expression.

U_1: [Chance] South are more likely to win than North (b: South). [Grade 6]

M_1: [Chance] It is more likely that North wins than they lose because there is 7 chances of them losing and 2 chances of them winning (b: South). [Grade 8]

R_1: [Chance] It means that North will have 2 chances to South's 7 chances to win the match. Usually it's x-1 but because x is not a whole number, they have to make the second number 2 to make the 1st number whole (b: South). [Grade 10]

Some students changed the context of expression over the years. The following student initially had no structure for understanding the numbers in the concrete symbolic mode, and considered the numbers as a date. In later years, the student identified the chance context, and then quantified this in a frequency ratio.

P: [None] Date. (b: Don't know). [Grade 6]

U_1: [Chance] North has a 7-2 chance of winning the footy game (b: The other team). [Grade 8]

U_2: [Frequency] They will win 2 of 7 games. Not much chance. (b: The other team). [Grade 10]

Similarly, the following student's responses changed from describing a score, to a chance expression, to a betting context involving money gained for a win.

IK: [Score] 7-2 means 7 goals to 2 goals. (b: North). [Grade 6]

U_1: [Chance] 7-2 means the chance it has of winning. 7-2 is an OK chance (b: South). [Grade 8]

U_2: [Betting] Is the odds of winning. If you were to put a bet on it, for every \$2 you put on it you would receive a \$7 profit (b: South). [Grade 10]

Another student built upon the concept of betting to later offer a chance expression.

U_2: [Betting] \$7 for every \$2 bet (b: Better). [Grade 6]

U_2: [Betting] If a person bet \$2 that North would win they would receive \$7 in return (b: No response). [Grade 8]

R_2: [Chance] This means that there is 7/9 chance of losing and 2/9 chance of winning (b: South). [Grade 10]

There were some students whose responses reverted to lower levels in later years. For the following student's responses, the third response does not clearly distinguish whether or not the return of \$7 includes the \$2, that is it does not clearly distinguish whether "7-2" is a part-part or part-whole ratio. Hence it is classified at the U_2 level.

U_2: [Score] 7-2 means that North has 7 points and South has 2 points. (b: North). [Grade 6]

R_2: [Frequency] If 9 games were played North would win 2. (b: South). [Grade 8]

U_2: [Betting] If North wins, punters will be paid out \$7 for every \$2 they bet on North (b: South). [Grade 10]

A summary of results for these four-year longitudinal data is shown in Table 2. At all grade levels, the modal response for females was to interpret the numbers as the score in the game, while males offered responses at higher SOLO levels, particularly using betting and chance contexts of expression. These differences favouring males resulted in significant differences of mean scores at Grade 6 ($t_{99} = 3.01$, $p = 0.0034$), Grade 8 ($t_{99} = 3.69$, $p = 0.0004$) and Grade 10 ($t_{99} = 3.15$, $p = 0.0021$). Both males and females improved from Grade 6 to Grade 8 ($t_{99} = 2.63$, $p = 0.010$), and from Grade 8 to Grade 10 ($t_{99} = 3.89$, $p < 0.001$), and there were no gender differences for improvements, nor differences in the improvements between the two two-year periods.

Table 2

Four-Year Longitudinal Comparisons of Percentage of SOLO Levels, Mean Scores, and Percentage of Contexts of Expression by Gender

SOLO Level	Grade 6 (1993) -		Grade 8 (1995) -		Grade 10 (1997)	
	Female	Male	Female	Male	Female	Male
NR/P	23	18	11	16	5	4
IK	61	31	55	11	48	16
U_1	7	24	13	27	5	7
M_1	4	11	14	22	16	22
R_1	-	2	-	5	5	11
U_2	5	13	7	16	14	36
M_2	-	-	-	2	2	2
R_2	-	-	-	2	4	2
Mean Score	0.36	1.07	0.70	1.71	1.41	2.51
Context						
NR/P (None)	23	18	11	16	5	4
IK (Score)	61	31	55	11	48	16
Betting	2	22	13	29	16	40
Chance	11	20	14	36	18	24
Frequency	4	9	7	9	13	16
N	56	45	56	45	56	45

Exploring further the changes in individual students' responses in 1993, 1995 and 1997, it was found that 34 out of 101 responses decreased in SOLO level over one of the time intervals. Of the 34 reversions, 13 were to other levels in the concrete symbolic mode, 16 to the ikonic score interpretation, and 5 to no response. In considering the dominant contexts used in individual students' responses, 28 students used more than one of the three contexts, 50 offered one context only (20 betting, 19 chance, 11 frequency; 39 included the ikonic score interpretation in at least one year, 11 did not), and 23 used no context (non-response, prestructural, or ikonic score categories). These results must be interpreted with respect to the high percentage using the ikonic score interpretation. In particular, use of one context only (50 responses) does not represent consistent use of the context in responses for each of the three years, but rather use of just one of the three context in at least one of the three years.

Table 3 shows students assessed longitudinally over a two-year interval for Grades 6-8, 8-10 and 9-11. Data for Grade 6-8 and Grade 9-11 are combined for the periods 1993-1995 and 1995-1997, as there were no significant differences of scores for responses or improvements between these periods. Data for Grade 8-10 from 1995-1997 are combined for both new Grade 8's in 1995 and those repeating from Grade 6 in 1993, as no repeat effects were evident. The table shows grade differences and gender differences at all levels ($p < 0.01$ in each case), except Grade 11 where females do not differ from males. Over the two-year period improvement occurred at all levels, with no gender differences except from Grade 9 to 11, where females improved more than males ($t_{143} = 2.27$, $p = 0.0247$), evident in Table 3 where mean score for males did not change (2.47 at Grade 9 to 2.49 at Grade 11) while females in the two-year period caught up to the male mean score (1.56 at Grade 9 to 2.38 at Grade 11).

Of the total 648 pairs of responses over two year intervals, 21% decreased in SOLO level, 39% remained at the same level, and 40% increased. For the contexts these students used over the two years, 18% used the same context twice (9% betting, 7% chance, 2% frequency), 18% used two of the three contexts, 24% used a context in the second year after using no context or the ikonic score interpretation in the first year, 11% changed from a context to no context or the ikonic score interpretation, and 30% offered none of the three contexts in either year. Of 303 students who used one of the three contexts in the first year, 76% subsequently used a context, 17% changed to the score interpretation, and 8% subsequently used no context.

Table 3
Two-Year Longitudinal Comparisons of Percentage of SOLO Levels, Mean Scores, and Percentage of Contexts of Expression by Gender

SOLO	Grade 6	-	Grade 8		Grade 8	-	Grade 10		Grade 9	-	Grade 11	
Level	F	M	F	M	F	M	F	M	F	M	F	M
NR/P	22	19	14	11	13	13	4	7	3	5	2	17
IK	59	38	51	25	48	16	47	21	43	20	16	10
U_1	5	18	14	18	11	25	10	13	9	5	10	3
M_1	5	14	13	21	19	21	15	21	22	19	31	25
R_1	4	2	2	9	2	7	6	9	1	10	10	7
U_2	5	8	7	9	5	13	16	23	13	31	15	24
M_2	-	-	-	2	1	4	2	1	1	8	10	2
R_2	-	-	-	1	1	2	2	5	7	2	3	12
Mean Score	0.45	0.86	0.71	1.34	0.84	1.69	1.38	2.08	1.56	2.47	2.38	2.49
Context												
NR/P (None)	22	19	14	11	13	13	4	7	3	5	2	17
IK (Score)	59	38	51	25	48	16	47	21	43	20	16	10
Betting	3	17	12	26	15	31	15	37	24	37	42	39
Chance	11	22	17	26	19	34	22	23	20	24	30	17
Frequency	5	5	7	8	6	6	12	12	9	14	9	17
N	133	125	133	125	124	121	124	121	86	59	86	59

Cross-Cohort Comparisons

Cross-cohort analyses compare the same grade level assessed in different years to monitor cohort changes, which may provide evidence of changes in effectiveness of curriculum implementation. Table 4 shows the comparisons of Grade 6 and 9 students in 1993, 1995 and 1997, and of Grade 8 and 11 students in 1995 and 1997. The score interpretation of the headline was again dominant at Grades 6 and 8, while Grade 9 and 11 students were more likely to offer responses at higher levels, using expressions in betting, chance or frequency contexts.

Table 4

Cross-Cohort Comparisons of Percentage of SOLO Levels, Mean Scores, and Percentage of Contexts of Expression, by Grade and by Year

SOLO	Grade 6			Grade 8		Grade 9			Grade 11	
Level	1993	1995	1997	1995	1997	1993	1995	1997	1995	1997
NR/P	19	29	17	15	14	9	15	6	11	6
IK	47	41	60	36	42	25	29	42	14	15
U_1	14	16	10	17	18	13	14	19	9	9
M_1	10	7	7	17	13	21	15	9	27	36
R_1	3	3	2	4	5	6	7	6	11	4
U_2	8	4	3	8	8	17	17	9	18	19
M_2	-	-	-	2	-	3	2	1	5	2
R_2	-	-	-	1	-	5	1	9	4	9
Mean Score	0.73	0.55	0.45	1.12	0.91	1.90	1.48	1.49	2.19	2.32
Context										
NR/P (None)	19	29	17	15	14	9	15	6	11	6
IK (Score)	47	41	60	36	42	25	29	42	14	15
Betting	12	5	7	21	15	24	26	14	36	43
Chance	16	23	14	24	22	28	22	31	27	21
Frequency	6	2	2	5	8	14	8	8	11	15
N	301	223	177	352	171	355	324	101	197	47

Significant differences between the mean scores for different years were found for Grade 6 ($F_{2,698} = 4.189$, $p = 0.0155$) and Grade 9 students ($F_{2,777} = 5.335$, $p = 0.0050$). In both cases the mean score of students in 1993 was higher than those in 1995 and 1997, indicating a reduction in performance over the years, as shown in Table 4. For Grade 8 and 11 students, the numbers of new and repeating students differed across the years, making interpretation more difficult. Using data from both years, a positive effect of repeating was found at Grade 11 (new students mean score 1.75, repeat student 2.48, $t_{242} = 3.169$, $p = 0.0017$), but there was no such effect at Grade 8. No differences between mean scores in 1995 and 1997 were found for Grade 8 or 11 students, irrespective of whether using the entire data or only repeating students in each year.

For the three contexts of expression, Grade 6 and 8 students most commonly used the chance context, followed by the betting context and the frequency context, as shown in Table 4. A similar pattern of contexts was used by Grade 9 students in 1993 and 1997, while in 1995, and for Grade 11 students in all years, the betting context was most common, followed by the chance context and the frequency context.

Cross-Sectional Comparisons

A cross-sectional profile of student development by grade level was created by pooling all data collected in 1995 and 1997, as shown in Table 5. The 1993 data, collected from Grade 6 and 9 students, were eliminated from this analysis because they were significantly different to that values in 1995 and 1997, as discussed above. Data at Grade 8 included both repeat and new students as no differences were observed between the mean scores of these groups. For Grade 10, only 14 students had never done the survey before, and data from these students were combined with those of students repeating the survey once or twice from previous years, as no effects of repeating were found for these groups. For comparison purposes with the Grade 10 data, all Grade 11 responses were included despite the repeat effect (see cross-cohort analysis above). Thus the results in Table 5 include all data except from 1993, with the proviso on interpretation that Grade 10 and 11 results may be inflated due to effects of repeating the survey previously. The table shows the score interpretation of the headline was dominant at younger grade levels, while older students responded at higher levels. Analysis of variance found significant effects for grade ($F_{4,1854}$ = 552.29, $p < 0.0001$) and gender ($F_{1,1854}$ = 94.08, $p < 0.0001$), but no significant interaction. Males outperformed females at all grade levels, mainly due to the larger numbers of females who interpreted the headline as a score. Mean scores for groups when responses interpreting the headline as a score are eliminated from the analysis are shown in the bottom row of Table 5. No significant differences between mean scores of females and males are found at any grade levels, indicating that there are gender differences for the interpretation of the headline as a score, but not for response levels once students appreciate the numbers refer to one of the three contexts of expression.

Table 5
Cross-Sectional Comparison of Percentage of SOLO Levels, Mean Scores, Percentage of Contexts of Expression, and Mean Scores Excluding Score Interpretations, by Grade and by Gender

SOLO	Grade 6		Grade 8		Grade 9		Grade 10		Grade 11	
Level	F	M	F	M	F	M	F	M	F	M
NR/P	24	23	13	16	13	14	4	6	7	14
IK	59	39	50	25	39	25	47	20	17	11
U_1	8	20	14	21	15	15	9	13	11	7
M_1	5	9	14	18	14	14	14	21	30	28
R_1	2	3	3	7	5	8	5	9	10	9
U_2	2	6	6	9	11	18	16	24	15	22
M_2	-	-	<1	3	-	3	1	1	7	2
R_2	-	-	<1	1	3	3	3	6	2	8
Mean Score	0.35	0.69	0.78	1.34	1.23	1.72	1.42	2.14	2.11	2.32
Context										
NR/P (None)	24	23	13	16	13	14	4	6	7	14
IK (Score)	59	39	50	25	39	25	47	20	17	11
Betting	2	11	12	26	14	31	15	35	36	38
Chance	14	26	19	27	26	22	21	25	30	22
Frequency	2	2	6	6	8	8	13	13	9	15
N	220	180	268	255	208	217	136	136	127	117
Mean Score *(excl. IK-score)*	0.84 n=91	1.14 n=110	1.54 n=135	1.78 n=192	2.02 n=127	2.31 n=162	2.68 n=72	2.67 n=109	2.55 n=105	2.62 n=104

Discussion

Longitudinal development was observed for student responses assessed over two-year and four-year intervals, with general improvement in SOLO levels, and increasing use of the three contexts of expression. These results suggest the developmental model shown in Table 1 is a useful model for assessing student development in understanding odds from upper primary school through the secondary school years. One test of the model is the number of reversions to lower levels in subsequent years. For the two-year longitudinal change, of 303 students whose responses involved a context in the first year, 136 (45%) reverted to lower levels, higher than might be expected. It should be noted, however, that only 62 of these (20% of the 303) reverted to lower SOLO levels in the concrete symbolic mode, while 23 (8%) reverted to a prestructural or non-response, and 51 (17%) reverted to the ikonic score interpretation. The latter may not surprising given that teaching odds in applied settings such as this newspaper headline is unlikely to be universal or consistent across the secondary school years. Some teachers express anxiety about teaching the topic and occasionally say they would not do so (Watson & Moritz, 1998b). Hence students might be expected to offer an ikonic response as an alternative interpretation more readily in the sporting setting than they would in more conventional mathematics settings.

Gender differences favouring males were generally observed, except at Grade 11. This follows similar findings favouring males for selected grades or questions for comparison of odds (Green, 1991) and for chance measurement (Watson & Moritz, 1998a). These differences were particularly in the use of the ikonic score interpretation, and after eliminating these alternative responses from the analysis, there were no gender differences in levels of concrete symbolic responses — that is, once females understood the headline was about odds, they interpreted it at the same levels of sophistication as males. This result highlights the importance of appreciating the context in which mathematics is set in the classroom. It would appear that females are disadvantaged until the later years of schooling by the setting of odds in a sporting context. The solutions to this dilemma are in the hands of teachers and curriculum planners.

The higher scores observed for 1993 than 1995 and 1997 samples at Grades 6 and 9 deserve particular attention. It is important to note that the same NUD•IST script was used, thus the categorisation method was consistent across years. The 1997 Grade 9 sample was reduced in number due to some schools not offering new students because of other time commitments. Thus a school bias for Grade 9 may have reduced results for 1997. If, however, the 1995 Grade 9 sample and the 1995 and 1997 Grade 6 samples were truly representative of students at those grade levels, it would appear that a reduction in performance has occurred in understanding of odds. Further analyses of other survey questions are awaited to see if this result is of more widespread concern.

Acknowledgment

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An arithmetic game with strategic components - playing procedures of primary school children

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The project *children playing arithmetic games with strategic components* concerns with the existence and with types of strategies used during playing. In a qualitative study with an interpretive background we have observed primary school children of different grades playing arithmetic games with strategic components. In this paper we shall present in detail the procedure of children of grade 3 playing an addition-game with strategic character.

Introduction

The value of using learning games in the primary mathematics classroom is widely acknowledged. Nevertheless, until now few empirical studies have investigated their effectiveness. Our project deals with research on existence and development of strategic facilities with younger children. For this purpose we use arithmetic games with underlying strategic components. The games are designed to promote an active and discovery-oriented arithmetical learning process. They allow children to discover relationships between numbers and patterns such as the commutative law of addition and multiplication. They also embody frequently neglected strategic components such as recognizing multiple ways of achieving a score. Optimal strategic behaviour requires far-sighted and deductive thinking and reasoning.

The structure of the games is always nearly the same and resembles those games which children know from playing outside school. The different games vary in the arithmetic operations - each game has only one operation - and in the tasks which have to be solved.

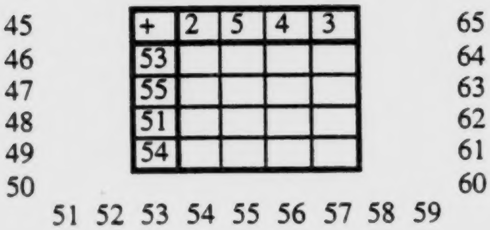
Normally the games are planned for two players. Concerning our interest on discussions about playing procedures we changed this situation. In our studies there are two teams playing against each other, each team consisting of two children and the teams sitting in different rooms. The communication about the proceedings of the game takes place by telephone.

Paradigmatically we watched one team very carefully by video-camera. We transcribed the video-tape and will present the transcript in full length in this paper.

The following questions can be formulated:

- do the children anticipate that there are several tasks with the same result,
- do they use mathematical rules when choosing a task,
- what are they mainly motivated by.

Presentation and Analysis of the game



+	2	5	4	3
53				
55				
51				
54				

The game consists of a matrix of tasks called *task field*. The tasks are determined by the matrix numbers and the given operation. The results of the tasks are arranged around the matrix. We call the numbers around the matrix *result field*. There are more numbers in the result field than can be achieved by calculating the tasks of the task field. To mark the elements in the task field the playing teams use tokens in different colours.

Playing rules:

1. Playing teams in turn place their tokens on free elements of the matrix, calculate the sum and place another token of the same colour on the correct result of the result field.
2. In case an element in the result field is already occupied, the playing team is allowed to replace the token of the opponent - or even its own token - by one of its own.
3. The game is over when all elements of the matrix are occupied. The winner is the playing team with the most occupied elements in the result field.

The tasks of the game are of such kind that they can easily be solved by children of third grade so that there would be enough capacity to discover mathematical relationships and rules and to plan strategic proceedings.

There are sixteen different tasks which yield only eight different results. At the end of the game there will be thirteen elements of the result field unoccupied.

Frequency of occupation in the result field

There are

- three elements which can be occupied only once - the numbers 53, 54, 60 - called *one-result elements,*
- two elements which can be occupied twice - the numbers 55, 59 - called *two-result elements,*
- three elements which can be occupied threetimes - the numbers 56, 57, 58 - *called three-result elements*.

Optimal strategic proceeding

The playing teams should first occupy the one-result elements in the result field. The three-result elements should follow in due course and they should avoid to be the first team to occupy a two-result element. When using an optimal strategy the team which starts in second will be the winner. To play in an optimal way they must not be the first team to occupy a two-result element. This refers also to three-result elements which are already once occupied.

Possibilities of variation

The game can be used in all classrooms of the primary school. There can be variations of

- size of the matrix
- operation
- tasks for calculation - less or more difficult-
- number of elements in the result field with multiple occupations
- frequency of results

Mathematical background of the tasks

The grade of difficulty of the tasks is rather low. There is only one task with result 60, all other tasks have results between 53 and 59, so that there is only once a change of the tens. These facts enable the children to develop strategies by using mathematical relationships and rules (see Table 1).

Result field	Tasks	Relationships and Rules	Frequency
53	51+2		One-result elements
54	51+3		
60	55+5		
55	51+4		Two-result elements
	53+2	53+2 = (51+2) + (4-2)	
59	54+5	Commutative law at the ones	
	55+4		
56	51+5		Three-result elements
	53+3	53+3 = (51+2) + (5-2)	
	54+2	54+2 = (51+3) + (5-3)	
57	53+4	Commutative law at the ones	
	54+3		
	55+2	Neighbour numbers compared to 54+3	
58	53+5	Commutative law at the ones	
	55+3		
	54+4	Neighbour numbers compared to 55+3	

Table 1

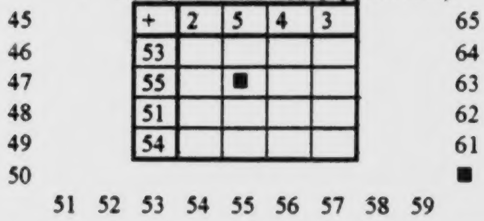
Transcript

The two boys - Sieg and Stev - whose playing behaviour we describe had as opponents another team of two boys. All four boys were third-graders. To give each team the possibilitiy of discussing strategies the two teams sat in different rooms and communicated about their moves by telephone. The opponents of Sieg and Stev started with the game. The tokens of Sieg and Stev are symbolized by ✖, the tokens of their opponents by ■. The leader of the inverstigation is called L.

Sieg and Stev knew the rules of the game since they had already played and won a similar game with subtraction as operation. At that occasion they had also been second to start.

Sieg: Who starts?
L: The other team.
Sieg: Ah, very good. Now we will beat them again. Is it important who is the better calculator?

1st move: 55+5 = 60 (opponents)



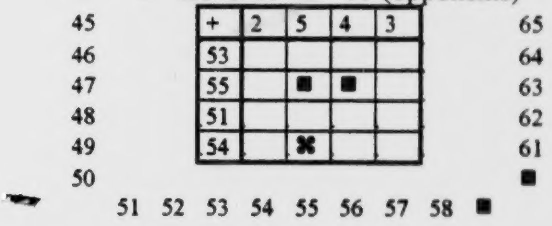
+ 2 5 4 3
53
55
51
54
45 46 47 48 49 50
51 52 53 54 55 56 57 58 59
65 64 63 62 61

Sieg: 60. We'll beat this.
Stev: 53, 54.
Sieg: No, 53+2. No.
Stev: Minus 5.
Sieg: Too bad. You can't beat them.
Stev: 55+5.
Sieg: 58, 60. Oh that's unkind of them. You can't beat this at this place. So we'll do also something unkind, so that they can't do anything. 54 + 5, you agree?
Stev: No. 53+2.
Sieg: Then they have 55. They can do it also.
Stev: How?
Sieg: Look, if they here (points to 55) plus 3. I tell you what they will do: they do simply 51+4. Then we do better 54+5 equals?
Stev: 59.

2nd move: 54+5 = 59.

Sieg: They won't throw us off any more. But they have an advantage.
Sieg: Now we must hope that they don't throw us off.
Stev: I don't think so.
Sieg: Oh dear they can do it yet. But they are not as clever as we are.

3rd move: 55+4 = 59 (opponents)



+ 2 5 4 3
53
55
51
54
45 46 47 48 49 50
51 52 53 54 55 56 57 58
65 64 63 62 61

Sieg: Bad boy. Wait, we also throw them off. 53+...No. 55 plus 2, 57, plus 3, 58.
Stev: How shall we do it?
Sieg: I don't know. They are well in advance. Now let's take the biggest task. 60, ok?
Stev: Is already.
Sieg: And then we throw them off. Not possible. Then we can't throw them off. Then let's take 55+2.

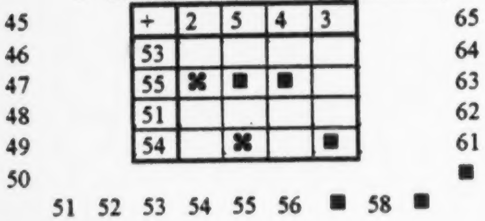
4th move: 55+2 = 57.

Sieg: Hopefully they can ...They can beat us again. Or? Yes they can (points to54+3). Then they catch us.
Stev. Anyhow we loose.

Sieg: That's not yet clear. There are still many fields to do. When they make a mistake and don't beat us then we'll win.

Sieg: We can't beat them. Not on 60 and not on 59.

5th move: 54+3 = 57 (opponents)



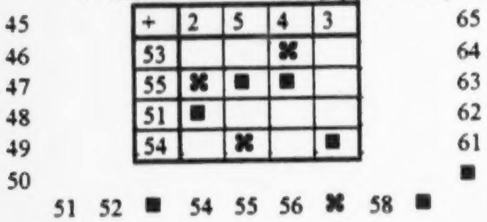
+	2	5	4	3
53				
55	✖	■	■	
51				
54		✖		■

Sieg: Oh, look. We'll beat them. Plus 5 (points to 53). No. Plus 2.Plus 4. Yes. Now away.

6th move: 53+4 = 57.

Sieg: They will be annoyed now. We can still do something. Now they can't throw us off that field. They make the first and we can do the second. If they do anything with 58 we can throw them away by 55+3. I hope they won't make 57.

7th move: 51+2 = 53 (opponents)



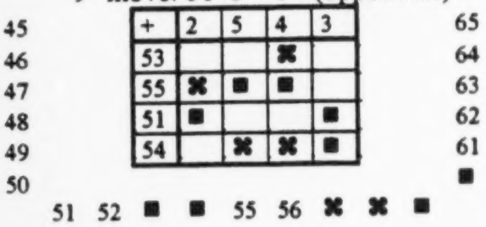
+	2	5	4	3
53			✖	
55	✖	■	■	
51	■			
54		✖		■

Sieg: So, can we beat them? 53? Let's take 51+...No. We don't have 50. 54. No. 54+2 = 56. Can they do anything with it? Plus 5 works as well. If we do 55+3 they can do 53+5. They can beat us nearly everywhere. No. 54 they can't (points to 51+3, then to 51+4). 55. Can they do anything with 55? No. They can still beat us. 56 also. No. 8...Let's do 58.

8th move: 54+4 = 58.

Sieg: Let's look what to do. Oh dear that also works (points to 55+3). I hope they won't do that.

9th move: 51+3 = 54 (opponents)



+	2	5	4	3
53			✖	
55	✖	■	■	
51	■			■
54		✖	✖	■

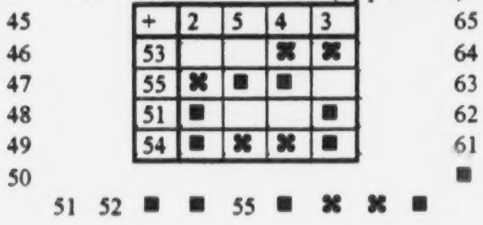
Sieg: Shall we throw them off? Let's take 51. No. 53. No. 54 plus...No. 51+3, ok? No, plus 4. 55. No, then they will throw us off. 58 is good, this is occupied by us.

55 or 56, ok?

10th move: 53+3 = 56.

Sieg: They beat us. Perhaps they do even 51+4.

11th move: 54+2 = 56 (opponents)



45
46
47
48
49
50
51 52 ■ ■ 55 ■ ✖ ✖ ■
■
61
62
63
64
65

+	2	5	4	3
53			✖	✖
55	✖	■	■	
51	■			■
54	■	✖	✖	■

Stev: Oh, no, no.

Sieg: We can take this away. Shall I say what we will do?

Stev: 51+ ...

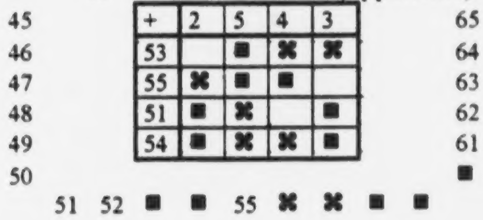
Sieg: Exactly. 51+5. Away with you. Now they can't do anything more.

12th move: 51+5 = 56.

Sieg: Now they can't beat us. Or? Plus 2 doesn't work. Plus 4 doesn't work. Plus 3 doesn't work. Yes, they can't beat. But perhaps they are in advance with one more.

Stev: No, we have all the same. They have six and we have six.

13th move: 53+5 = 58 (opponents)



45
46
47
48
49
50
51 52 ■ ■ 55 ✖ ✖ ■ ■
■
61
62
63
64
65

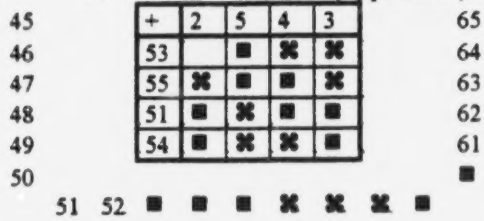
+	2	5	4	3
53		■	✖	✖
55	✖	■	■	
51	■	✖		■
54	■	✖	✖	■

Sieg: Can we throw it off? Let's think. Let's just do 55+3 = 58.

14th move: 55+3 = 58

Sieg: Now they can't do anything more. And you said we would loose (to Stev). If they do 53+2 = 55 then we will win, no deuce. The result will only be 55 (points to 51+4). Then we will beat them.

15th move: 51+4 = 55 (opponents)



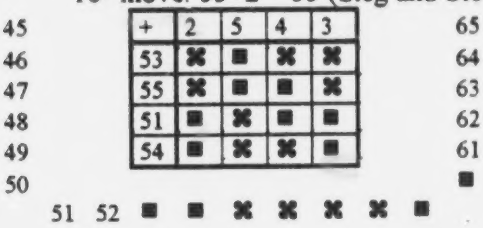
45
46
47
48
49
50
51 52 ■ ■ ■ ✖ ✖ ✖ ■
■
61
62
63
64
65

+	2	5	4	3
53		■	✖	✖
55	✖	■	■	✖
51	■	✖	■	■
54	■	✖	✖	■

Sieg: Perhaps we will still win. So, and now we put it there (points to 53+2).

16th move: 53+2 = 55.

16th move: 53+2 = 55 (Sieg and Stev)



Sieg: We won.
Stev: One, two, three, four, five, six, seven, eight.
Sieg: Did we win?
Stev: No. Deuce.
Sieg: One, two, three, four. Deuce. Good as well.

Results and Conclusion

The two children are mainly motivated by removing the opponent's token from the result field (e.g. before the first move of the opponents "...we will beat them again."). Immediately after each move of the opponents they make remarks like "we'll beat this" or „we can throw this off".

There is no evidence that the two children had planned their moves at the very beginning of the play. There are no indicators that they have recognized the optimal strategy, they never occupy a one-result element Their main strategy is the immediate removing of an opponent's token from the result field whenever this is possible. This strategy is very predominant. In the moment of the move it prevents them from thinking of their own token being eventually removed afterwards by the opponents. Nevertheless they show far-sighted behaviour in an initial stage by thinking about consequences when their move is done. This happens after each of their moves. But they also recognize a two-result element and its disadvantage in advance (discussion after the first move (opponents) about the two-result element 55). We could not find that they also recognized a three-result element in advance.

The use of mathematical rules is shown in the discussion after the seventh move (opponents) where they refer to the commutative law "5+3 = 3+5".

The game gives a lot of opportunities to practice calculating. If not in the first run the children could develop and improve far-sighted thinking by playing the game several times.

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Ability Grouping: Some Implications for Building Mathematical Understanding

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In response to an interview question on features of classroom organisation that support the development of mathematical understanding, all four teachers interviewed referred to the benefits of "ability grouping". This was seen as a means of structuring learning conditions so that teachers could attend to children with similar levels of understanding. Subsequently, achievement-grouped classes were videotaped. Analysis of the data showed that interactions within this mode of organisation demonstrated the same problematic features as whole-class teaching. It is concluded that if achievement grouping is to be used, other strategies need to be used to make mathematics pedagogy inclusive.

Introduction

In a research project entitled *Teachers' constructions of their roles in building mathematical understanding*, four teachers from one school were videotaped teaching about six lessons each. Throughout this observation period, interviews and less informal discussions with the teachers were recorded. The results of this stage of the project will be used to inform the construction of a survey instrument researching the notion of mathematical understanding, to be used with a wider population of primary mathematics teachers.

During the interviews held with the teachers in the first few weeks of their observation periods, each was asked to "... talk about the different ways (they) organise the classroom that seem to facilitate the development of children's mathematical understanding". This paper focuses on what was the first claim made by three of the teachers and at a later of the interview with the fourth teacher. In response to this question, all four teachers claimed that the use of ability groups allows teachers to focus on children's understanding because the children in such groups have "common levels of understanding" (Teacher R, October 1997).

This research report presents some of the teachers' claims and comments about ability grouping, a summary of the results of this aspect of the videotape analysis, and some data to represent a fairly typical pattern of interaction in these groups. It concludes with a statement about the implications for teachers of this piece of research.

Ability grouping

How classroom participants act is largely a product of their theories about learning and teaching, about the subject matter involved, and about the roles individuals should play within the social contexts of schooling. Teachers' ideas about ability grouping in mathematics classes are a product of their personal histories as students and as professionals working in an environment where underlying assumptions about pedagogy frequently remain unchallenged. The notion of *ability* is rarely portrayed as problematic, and the teachers' right to set up various forms of ability groups is often actively encouraged by administrators and parents, and rarely questioned. Nevertheless, most administrators, teachers and even students are aware of some of the potentially undesirable effects of ability grouping, and in practical contexts they are aware of the tensions created. For instance, the conflict between the educational goals of excellence and equity is a significant problem in mathematics teaching, and the tensions surrounding ability grouping are one of the manifestations of this larger conflict. Amongst other teaching facets, Sullivan (1996, p. 1) identified a tension that he called "educating for the best <—> educating for everyone", just one of the many excellence–equity dilemmas that make decisions about curriculum planning and classroom organisation problematic.

The literature on ability grouping is extensive, and much of this research has been carried out in mathematics education because mathematics is used as a mechanism for streaming students into particular courses and occupations, and hence it is the most commonly ability-streamed subject in primary and secondary schools (Loveless, 1994). Since the 1960s, there has been a continual stream of evidence to show that within-class grouping has little effect on achievement in mathematics, and given the likelihood of loss of self-esteem in slower groups the argument is generally made that ability grouping and/or streaming should not be used. Recent studies taking this form include those of Bode (1996) and Fuligni (1995). Similarly, some studies have shown that the use of fixed ability groups led to lower achievement than a whole-class model (for example, Mason & Good, 1993). Some researchers have criticised ability grouping in mathematics classes because it has been shown to increase the gap between students through exposure to limited models, or because it encourages differences in teachers' expectations thus leading to tailoring of instruction (see for instance Cahan, 1996).

However, the implications of research finding in this area are not unambiguous. If one is to take an "educating for the best" position, it is clear that there are some potential advantages to ability grouping, and for this reason the right of schools to stream students at secondary levels is rarely questioned. In fact, some extensive studies, such as that by Brewer (1995), have shown that long-term streaming leads to increased performance for uppertrack groups—although the research in this area rarely gathers data on, or even questions, the social implications of such practices. Brewer noted the social pressure on schools to accept the idea of grouping by ability, collecting evidence that in schools where this style of grouping is common, the students, teachers and parents generally fear that achievement would be reduced if streaming were to be abandoned.

Shorter-term and more flexible ability grouping is also seen to bring benefits to all children (Numeracy Task Force, 1998). For instance, if group size is adjusted according to the apparent need for individual attention, teachers can spend more time working with lower-achieving individuals, and they are not so likely to have teaching that is directed at their peers go "over their heads". Similarly, the better students can be challenged and coached together, benefiting from higher-level interaction with their peers and the teacher; and middle groups need not either be daunted by work that is challenging for children with higher levels of understanding or be held back while lower achievers have their needs met. The Numeracy Task Force report, published only recently in England recognised such potential advantages when it recommends that:

> One purpose of group work is to allow a manageable degree of differentiation around a common theme. Groups can be organised by attainment, and while the main body of the class works on a set task, a more challenging task can be set for the most able, with a simplified task for those who would benefit from this. (Numeracy Task Force, 1998, p. 20)

However, as with all of the key dilemmas in mathematics education, there are no simple solutions to the conflict between teaching for excellence and teaching for equity. There is a necessity to investigate what happens within ability groups, and to gather data on whether assumptions such as "common levels of understanding" can be supported.

Methodology

The teachers in the school that provided the context for my research work as pairs in planning, teaching and assessing mathematics. Four teachers (two Year 2, and two Year 6) were the focus of this project. The general aims of the research were to identify (a) what the teachers claim to do to build their pupils' mathematical understandings, as well as what they actually do; (b) the different types of understanding required of primary-school mathematics students; (c) some personal, institutional and socio-cultural factors that support or constrain teachers' work in developing mathematical understanding; and (d) how success (or lack of success) in this field is recognised.

About six lessons taught by each teacher, and some jointly-taught lessons for each pair, were videotaped. Casual discussions were held with teachers after most of the lessons, and more structured interviews were held midway through each observation period as well as at a later date when some of the data had been analysed. The discussions and interviews were audiotaped.

To date, the videotape analysis has aimed to identify four distinct components of the interaction, although it is recognised that these factors worked interactively. The first area of focus has been what the teachers did to develop their pupils' understandings of particular concepts being taught. Here, the teacher actions studied have included components such as patterns of explanation, types of questions and responses, the use of metaphors and examples, and demonstration with gestures, diagrams and aids, etc. A second level of analysis has focused on how some child exhibited what seemed to be a growth of understanding, with a particular emphasis on "breakthrough" episodes, as well as different types of understanding exhibited (e.g. understanding of explanations c.f. understanding of mathematical concepts; instrumental versus relational understanding [Skemp, 1976]). Teachers' and children's body language, oral exchanges, use of concrete materials, written work, children's explanations, their responses to the teachers' questions, and other similar indicators of the children's understandings have been noted. Third, broader aspects such as modes of classroom organisation and the dynamics of the interaction that seem to impinge on learning of mathematics concepts, including social interactions that seem to support or inhibit the development of understanding have been examined; with particular focus on the experiences of individual children. Factors noted here have included prior understandings that different children demonstrate early in the lessons, the ways in which they attended to or ignore the teaching and learning activities, the amounts and types of individual attention and feedback each child received, and so on. The fourth category of focus has consisted of phenomena taken from the literature on what teachers can do to develop understanding, such as the "folding back" technique expounded by Martin, Pirie and Kieren (1995) that developed from a "dynamic theory of mathematical understanding" (Pirie and Kieren, 1991), metacognitive experiences (e.g. Greeno & Riley, 1987), relationships between understanding and mathematical processing (e.g. Arnold, 1997), and qualities of understanding (e.g. Skemp), as well as constructivist and socio-cultural perspectives on the development of understanding.

The focus on ability group in this section of the research project was not part of the original research design. It was identified as only one of many potential areas of engagement related to the third focus above, when each of the four teachers mentioned in the first formal interview that this mode of classroom organisation makes it easier for teachers to build their pupils' mathematical understanding. Subsequently, the tapes of the four lessons (three Year 2 and one Year 6) where children had worked in ability groups were re-analysed in the light of this claim.

In each of the four achievement-grouped lessons, the Number area of the curriculum was central. Two of the Year 2 lessons and the Year 6 lesson were on place value, and the remaining lesson taught with ability groups was a Year 2 lesson on problem solving using multiplication. In all cases, the groups taught by the classroom teacher were small (5-6 pupils in each of two more able and two less able groups being videotaped), while the rest of the class worked in groups under the direction of other people—other teachers, parents, or child monitors—in the same room, other classrooms, withdrawal areas, or outside. I did not choose the groups or children to film, because I was "following" each of the classroom teachers in turn.

The school context in relation to ability grouping in mathematics

Ability grouping is used throughout the school, but it is school policy that students not be "tracked" in longer-term streams. Typically, ability groups are used only once a week, and the teaching focuses on a particular concept. The groups are temporary, lasting

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from one to three weeks; and are created by the teachers on the basis of diagnostic tests and/or their perceptions of individual students' knowledge and skills as the pupils complete regular seatwork. The groups do not have names—a typical instruction to form new groups is "F,R,L,C, ... and R; get a bead frame and work with Mrs Smith, please". If the same groups continue for several mathematics lessons, the teacher will call up, "Those people who did bead frames with Mrs Smith", and so on. Thus groups are extremely flexible, with, for instance, no achievement-grouped lessons in one week, several over the next few weeks, etc. Groups dissolve as the class moves on to another topic or the concept is explored further in a whole-class situation.

The teachers' ideas about ability grouping

The following three quotations from transcribed discussions are typical of the reasons given for the use of achievement grouping:

> Teacher B: (After place value lesson 1, Year 6) There are some bright kids here who really do not get extended in ordinary lessons, and some slower ones who get left behind. This works well at both ends. But not all the time—it would do more harm that good if we used it all the time. (Question: Why?) Two things really. One is that the slower kids would not progress, because they learn a lot from the others and they would never be challenged by listening to higher-level ideas. It would not be good for them socially either. Kids this age do not need to be labelled "smart" or "dumb" by being stuck in a particular group. There is already enough of that at this age.

> Teacher W: (After place value lesson 2, Year 2) Kids like M and S get left behind in a grade situation. I just can't spend enough time with them. They are really weak. They need lots of time on really basic stuff. Even counting—you saw M yesterday. I really don't think she learns anything most of the time because she has not got the underlying concepts. In a small group at least I can give her time and really get down to her level.

> Teacher W: (After problem solving lesson 1, Year 2) I wanted to do problem solving in groups because that lot never gets extended. They really fly, and learn a lot from each other when they are together—like L did today. She is so bright, she needs some challenging work. Nearly all of what we do as a grade, she already knows. She works so fast—too fast because she is so keen to finish that she makes mistakes. She needs work like this to get her thinking and to teach her that maths is about nutting out real problems, not just flying through routine work and getting it all right.

In summary, the teachers studied used ability grouping with the interests of the children in mind, and saw it as a means of catering for individual levels of knowledge and skill—and hence as a way of creating opportunities for enhanced development of mathematical understanding. Given this context, I was interested to analyse whether the achievement grouping did seem to serve these purposes for all, or at least some, children.

Results of video analysis

Analysis of the videos of interactions in the achievement-grouped classes showed that many of the "excellence <—> equity" issues present in whole class teaching also arose in these smaller groups. Space does not allow a representative sample of interactions to be displayed so I present one snippet of transcript as a context for the discussion to follow. This snippet was selected because it portrays many of the features noted in each of the four achievement-grouped lessons filmed.

Lesson: Year 2, Place value 1. The weakest of five groups was working with the teacher. The five children had been making groups of ten with counters and had been asked to record "one group of ten and two ones" on the whiteboard. Two of them had written it in words, but the teacher was aiming at the "number sentence" type of representation, or at least "12" so that she can point to the ten and the ones.

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Speaker	Dialogue (edited)	My comments
Claire	I've thought of it. (takes the pen; starts to write "Ten and two ...")	
Teacher	Mmh. ... No, without the words. Can you write it with the numbers. Just without the words, but using the numbers. ... No, I don't want you to draw the picture, I want you to write the number for me, of what that is. It's (said with pauses between words) one group of ten; and two ones. How can you write that with just numbers? (Stephen raises hand quickly)	Sees that Claire is still thinking in terms of words, so stops hinting and gives specific directions, using repetition for emphasis.
Stephen	I know!	Claire had written 10 + 2 earlier, and had drawn the correct MAB representation, but she does not understand the idea of "one group of 10 and 2 more". The problem seems to be the "groups" idea/word.
Janelle	Eleven. (Claire has drawn a circle with two little lines in it.)	
Claire	It's one group with two in it.	
Teacher	One group with two in it. No, it's one group of *ten*. If I do this on the board ...(teacher draws a multiplication sign) ... what does that mean, that cross. What does that mean? (Claire does not respond.) What does that mean Janelle?	
Janelle	It means ... (pauses).	Teacher presents what she had expected (use of x symbol, leading to lesson aim of 1 x 10 + 2 x 1 representation)
Teacher	What's that sign there? What's that mean?	Janelle does not understand the meaning of x ...
Janelle	A cross.	
Teacher	It's a cross, but what does it mean?	
Claire	I know.	... and guesses. It also seems she does not know the meaning of the equals symbol.
Janelle	Equals?	
Teacher	No, it doesn't mean equals.	
Dianne	Plus?	Janelle and Dianne demonstrate they are not ready do understand what the lesson is aimed at teaching (representing tens and ones as symbols).
Teacher	It does not mean plus ... (teacher looks at Claire).	
Claire	Times?	
Teacher	Times. What else does it mean? (no response) Groups of! Or lots of!	
Stephen	I can do it.	
Teacher	You can do it? Oh, Stephen, you are working so well today. (Teacher gives Stephen the pen).	Claire knows name of symbol, but perhaps not its meaning.
Claire	I know another way.	Teacher probably assumed Stephen has taken her hints.
Teacher	What?	
Claire	Four, plus four, plus four.	Claire is confident with plus. Did she link times with repeated addition?
Teacher	Four plus four, plus four. Good girl! That's a great way of doing it.	
Dianne	I know another way.	
Teacher	You know another way of doing it? What?	
Dianne	You get ... you do a circle and you put ten in it.	
Teacher	Yes. Exactly what Stephen has just done. (Stephen has drawn a	
Dianne	circle with ten little lines in it.) You've got one group of ten. Good	Stephen's representation is not what the teacher expected.
Teacher	boy. This is actually what Stephen has done and this is what Janelle has done, and Claire. You've all done one (writes figure 1, long pause; writes x).	Stephen knows x means "times" but does he think of it as "groups of"?
Stephen	Times.	
Teacher	Put them down (points at Diane who was playing with counters) ... one ... leave them, look ... (points to x) group of (pauses) ten (writes ten, pauses) and (writes +).	Teacher presents what she was aiming at, but lack of understanding from Dianne is obvious. Stephen is the only one attending.
Claire	Plus.	
Teacher	Two (writes 2) ... groups of (writes x) of (writes 1)...	
Dianne	(incorrectly pre-empting what teacher is about to write) Ten.	Diane watches teacher writing but has not followed any of the preceding exchange.
Teacher	One!	
Dianne	One.	

In the next section of the lesson (about fourteen minutes later), the teacher focused on Stephen, who seemed to be closest to understanding her aim—to teach symbolic recording of tens and ones. He remembered "times" and could represent a group of ten. The following interactions took place after the group had explored twelve then fourteen, and after the teacher had asked all of the children to make "one group of ten and three ones".

Laura	I've got it, Mrs W...	Laura has made a row of thirteen counters. Perhaps she doubts this is correct, as Stephen and Claire have put out two groups each (10 + 4). Perhaps she does not understand what the teacher wants her to say in repose to "What have you made?"
Teacher	Good. Show me. Can you show me where your group of ten is? (child points to a long row of 13 counters) Where? 'cause they're all joined up. Can you show me which is your group of ten? (Laura points to first ten, indicating a small gap that follows them) Oh, I see, that's your group of ten and those three ones are separate. They're in another group by themselves. What have you made? (Laura does not respond, after waiting for a few seconds, the teacher attends to Stephen who has fourteen counters out.)	
Laura	I know what I made.	Laura has thought of an appropriate response.
Teacher	(to Stephen) Two groups of one ...	
Laura	I know what I made. Mrs W..., Mrs W...	
Teacher	(to Stephen) No, I said to make a group of three leftovers, so you only need three.	
Laura	Mrs W, I know what it is.	Was this just a slip? Perhaps not—he never understood the idea of "groups of one".
Stephen	Three groups of three.	
Teacher	It's not three groups of three.	
Laura	I know what it is.	
Teacher	One ... That's one group of one ... one group of one, two groups of one ... if you put all those three together, that would make a group, and what would that be? How many groups have you got now?	Stephen does not seem to understand that understand that three counters can be called one group of three. (Their recent work with groups has been with MAB blocks, where the only groups are of ten.)
Stephen	Three.	
Teacher	You haven't got three groups. You've got ... ? If I had them like that (spreads them out) , now you've got three groups.	
Laura	I know what it equals (loud, moving forward on knees, hand up).	
Teacher	(to Stephen) And how many are in each group? How many's in this group? How many's in this group? How many's in this group?	
Laura	I know what it is (softly, sitting back on heels).	
Teacher	(to Stephen) That would be three groups of one, but if I bunch them all up together and make a little group, like this, in my hand, you're got one group of ... how many? (no response) ... three. One group of three.	Teacher answers her own question. Does he understand?
Laura	Mrs W..., I know what it is (bending forward).	
Stephen	I get it.	
Teacher	You get it! Oh, good on you. How many's this? (ten counters)	Teacher expresses relief, but what does he "get"? Teacher has changed the focus of his attention from the three to the ten.
Stephen	Ten.	
Teacher	If I separate them all, like this, have you still got one group of ten?	
Stephen	No	
Teacher	No, what have you got? (long pause, no response; pushes the ten counters together again) How many?	
Stephen	One group of one. Two groups of (hesitantly) two ...	Stephen is really struggling with the "groups" language.
Teacher	Two groups of? They're still ones.	Models teachers' counting of ones (probably what he "got" previously); but not the expected answer because he is splitting up the group of ten.
Stephen	One. Three groups of one, four groups of one, (continues[inaudible])..	
Laura	Mrs W..., I know what it is now.	
Teacher	Nine groups of one, ten ...	Teacher emphasises the "group" idea that is needed to understand place value. Stephen understood this earlier,
Laura	Mrs W..., I know what it is.	
Teacher	Yeah, ten groups of one. If you put them all together, in a little	

	bunch, in a group, (emphasises each word) one group of ten.(turns to Laura) Show me your one group of ten and your three left over. What is it?	but does he now that he has started seeing teen a "groups of one"?
Laura	Thirteen.	Laura's demand is attended to. She has not realised the teacher is seeking a "groups of ten" response.
Teacher	Thirteen! Good. (to Dianne) Did you know it was thirteen?	
Dianne	Thirteen.	

At the end of the lesson, Stephen appeared to understand, although close analysis of the transcript and video actually gives little evidence of this. While other pupils were floundering, unable to grasp the "groups of ten" concept, Stephen certainly gave the impression that he had grasped what had been explained, and the teacher gave him some positive feedback.

Teacher	Two. So you've got ... one group of ten ... and you've got two left over. You've got one group of ten, right? And you've got two left over. How many have you got all together?
Janelle	Um.
Teacher	Count them all together. You've got?
Claire	&(simultaneously) Twelve. Twelve.
Laura	(teacher points to three children, each of whom says "Twelve"; turns to Janelle; points to Janelle).
Janelle	(Not looking at counters, copying the girls ho called out) Twelve.
Teacher	Twelve! Can you count your up and make sure? (Janelle starts counting.)
Stephen	That was easy.
Teacher	Oh, I'll have to think of a harder one for you next time?
Janelle	It's twelve.
Teacher	Very good. Okay. (looks directly at Stephen and smiles) Very good. (gives directions for packing up)

Of course, this teacher may be accused of allowing a boy to dominate her attention, and that criticism would be justified for this latter part of this lesson. However, in other lessons taught by the same teacher this was not the case, and boys were left out of the teaching while a girl who seemed to be taking a real interest in the new concept became the focus of attention. What seems essential to note is that these snippets are representative of a broader problem (not a necessarily a gender one), as follows.

Analysis of the four tapes of ability-grouped lessons, followed by a focused re-visiting of the other tapes, showed that in lesson after lesson, teachers responded to and came to concentrate their attention on, the child or group of children who seemed closest to meeting the teachers' aims. This was apparent for significant sections of lessons taught by all four teachers, with a variety of maths topics, and using any pattern of grouping or class organisation, and whether the teachers were attending to lower- or higher-ability or mixed groups of pupils.

Typically, the pupils who attracted attention came to the learning activity with what seemed to be the highest level of background knowledge, appeared to listen well, and gave indications (body language, responses, etc.) that they were understanding the instruction process and/or content—even if this were not the case. These children tended to give the teacher positive feedback with smiles and indications of a willingness to "have a go", and in contrast with the others were not distracted by the wider classroom environment or by the presence of manipulatives to play with. In short, they appeared engaged, and this engaged teachers.

These children were not necessarily the "stars" of the class—they varied with topics, grouping, etc. As in the lesson above, it generally took some time to identify them, and to begin to give them a disproportionate amount of teaching time and energy. Then the teachers worked hard with these students to, as socio-cultural theorists would say, pull them forward in the zone of proximal development (Vygotsky, 1962), possibly in an attempt to meet the teachers' aims for the activity rather than because of any conscious effort to develop one particular child's strengths.

After the lessons, whenever teachers were available for short discussions with me, a standard question I asked was "Which children seemed to understand (the concept the teacher had just articulated as his/her aim for the lesson or activity)?" In most cases, the teachers named children who had exhibited the behaviours outlined above. When asked how they knew that these children had understood, teachers actually mentioned behaviours easily identified in the videos (such as listening, responding, seeming confident) rather than getting work right or answering questions correctly. Thus the teachers' judgements about understanding were dependant on behaviours not necessarily linked with or correlating with the understanding of mathematical concepts.

Some implications of these findings

A study of only four teachers, and then only four ability-grouped lessons amongst the many that were videotaped, does not present enough breadth to make claims about this mode of classroom organisation. However, as limited as the research context and the data are, the findings of this part of the research project have important general implications for teachers.

First, it cannot be assumed that any group of children, no matter how small and no matter how they are grouped, have similar understandings of (a) instructions, or (b) mathematical concepts and processes, or (c) the individual and combined words and symbols we use to represent mathematical ideas. This brings into question the ability of teachers to plan lessons in full ahead of time unless very open and adaptable questions and conceptual contexts are employed.

Second, organising children into small groups based on what seems to be similar ability does not, in itself, allow teachers to develop the mathematical understandings of each child—inclusive practices, awareness of individual differences, and a spreading of any teacher's attention are still essential. There can be no assurance that skills mastered in one context (such as groups of tens with MAB) can be transferred into other contexts (such as groups of one with counters), or that every child will be able to make these links when they are explained and demonstrated by the teacher.

The next two implications are linked. Third, the aims that any teacher has for a lesson, or even one learning activity, are likely to be easier for some children to guess or grasp—perhaps because of their stronger procedural and/or content knowledge on entering the activity. Fourth, there is a danger that teachers will get more positive feedback from these children and hence will shape the lesson around their particular needs, subsequently giving only those individuals a feeling of achievement.

Fifth, there is a broader implication of this research—although I have not analysed any of the data with this point in mind—that certain types of pupils are most likely to exhibit the positive behaviours that attract teachers' attention. These include children who are overtly confident—a characteristic that is often linked with elements such as gender, sociocultural backgrounds, maturation, language ability, and self esteem arising from past successes (and the interaction of such factors). They also include children who are "good at maths". Such patterns of interaction would only serve to reify these advantages and to reinforce the disadvantage of others.

Conclusion

It has not been the aim of this paper to criticise the practice of ability grouping *per se*. In fact, it is likely that the pattern of usage observed has its advantages. For instance, it is possible that Stephen would have been likely to be left with less positive feelings after working with a more heterogenous group. Neither has it been my intention to find fault with the way that this form of grouping is used in a specific school or by particular teachers. It is clear that the school and teachers involved in the study use achievement groups temporarily and flexibly, with a knowledge of the potential effects of more fixed arrangements as well as with the interests of all students at heart. However, I do wish to raise the awareness of teachers and teacher educators in relation to the precarious nature of the assumption that commonly provides a rationale for this mode of classroom organisation: that there is a necessary connection between ability grouping and the development of better mathematical understanding.

A broader issue is that if the above pattern of interaction is present with small "like-ability" groups taught by experienced teachers who have good mathematical knowledge and excellent classroom organisation skills (that is, with the best-case scenario), then it is also likely to be present with other pedagogical contexts that are less controlled. It is clear that teachers need to be aware of the need to draw out and build on the mathematical understandings of children within different forms of classroom organisation, and to be trained in ways of doing this. At least, this study implies the need for teachers, teacher educators, and researchers to work together to develop more inclusive practices, no matter what form of grouping is used.

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Ethnographic Research in Mathematics Education: Using Different Types of Visual Data Refined from Videotapes

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Some ethnomethodological research techniques that can be used to analyse social activity in mathematics classrooms are examined. A method where sections of video-audio tapes were transcribed to facilitate detailed inquiry, and individual "photographs" were used to reproduce some of the data not available in transcriptions, is described. Snippets of the original data were also available in movie format. This paper outlines some of the advantages and limitations of each of these three forms of visual data, and raises issues of anonymity and subjectivity.

Introduction

The use of video technology in mathematics education research is well established. It is used as a tool to capture classroom action, from which examples of activity type can be identified and discussed (see for instance, Barnett, 1991; Frid, 1994; Frid & Malone, 1993). Research data in video format is also integrated with other data in various ways (see Clarke & Kessel, 1995; Lambdin, Duffy, & Moore, 1995; Maher, Pantozzi, & Martino, 1996). It also enables events in mathematics classrooms to be used to stimulate recall of events (Owens, 1990). At the 1997 MERGA conference, I reported on how video is being used in a current research project to capture classroom interaction and to enable its close analysis (see Mousley, 1997). I noted that videotapes have some advantages over field notes and audiotapes, especially because they enable the rich and complex data to be revisited many times in order to be examined through different lenses and for various purposes .

This paper focuses on my later use of the videotapes, when snippets of video that were judged to be potentially useful were converted to two extra forms of visual data: transcripts and photographs. The paper provides a brief report of the processes used and then undertakes a comparison of the three different forms of visual data.

Conversion Processes

Videos of classrooms all share one characteristic—they offer us a wealth of detail. However, while opening up opportunities for thorough analysis of classroom events, the very richness of the data presents a problem. Spindler & Spindler (1992) articulate this well:

> No one can sit in a classroom and take notes on more than a very few of the things that come to one's attention simultaneously, and questions always arise as to why one's attention was focused in a given direction and not others. But the problem with much of what collects using [video] recording is that these data are so complex and inclusive that one has, in effect, to do the field work over again in order to analyse the material thus collected. (p. 77)

Converting video data to other formats can be a useful part of this revisiting process, in that each of the linked formats (audio only, video only, photo frames, and transcripts) serves to contain a researcher's focus and hence encourages reflection on a more limited field of activity.

After the original videotapes were digitised and compressed, the data existed as documents on a hard disc; each one a digitised "movie" of a full lesson. Using predetermined and emerging categories, each snippet of video that seemed relevant to the research question and its literature was indexed according to the event type and entered into a spreadsheet. That is a link to it was pasted in to the sheet, just as one would paste a graphic into a word document; and clicking on its icon plays the relevant snippet. For each snippet I also typed a note on its origins, a note on its categorisation, and a short descriptor (see Mousley, 1997). During this process, and as my analysis progressed, possible uses for different sections were noted in a "diary" column on the spreadsheet, along with questions to ask the teachers in further interviews, items that seemed to relate to points arising from the literature review or to other data, the need to seek further examples and non-examples of the event types, and so on.

To date, through this process, many sections of the lessons have been subjected to close examination in relation to "mathematical understanding", the topic of my thesis. One step in this analysis involved transcribing that section of the tape, for reasons outlined below. Another was to convert selected frames from the videotape to "photographs". The process used here involved (a) choosing suitable frames, (b) saving them as freeze-frame images, (c) opening them using a graphic software program, (d) adjusting the photos' contrast, size, etc. to clarify the images and focus on the important action, and (d) saving them in a format that allowed them to be pasted into relevant sections of the thesis and appendices.

Transcription

The chief reason for transcribing sections of the data was that PhD theses are still generally expected to be print documents that are examined in hard-copy format and then become part of a shelved library collection. The regulations that exist in most universities (expected length, presentation, etc.) do not even recognise the possibility of alternate forms of presentation for the thesis itself, although performance and exegesis are accepted as conjunct forms of presentation, and audiotapes are now more commonly accepted as appendices. Thus it was necessary to have examples of classroom interaction that could fit within this traditional format.

The second reason for transcribing pertinent sections of the video-audio tapes was to provide a basis for close analysis of the data. The transcription process, while tedious, has proved invaluable. It has brought aside comments, difficult to interpret responses, and other features that one does not generally focus on as part of an aural experience into a form that has promoted more careful analysis and reflection. For instance, as I typed a child's response to a teacher's question, my mind would wander to possible meanings that the child could have held and may have been trying to express. Did the boy who said almost inaudibly that, "Half an hour is sixty minutes" make a simple mistake and really have the necessary knowledge, or was he thinking ahead in the problem to the fact that it required the use of minutes from more than one half hour (two hours, in fact), or did he really believe what he had claimed? On first viewing of the lesson (through the lens of the video camera), and even on re-viewing the tape, the action I was watching was so fast and so complex that I did not have time for such considerations, and in fact I am not sure that I even heard his claim. However, the act of and pace of transcription (or perhaps the tedium of transcription) gave me the opportunity to remark on such comments, and prompted me look back and forward to other parts of the lesson for evidence of the meaning that that child held. This also led to a closer analysis of how the teacher appeared to assume one specific meaning and how this assumption affected the way she dealt with his comment at a later stage of the lesson.

In general, I have found that transcriptions are useful in that they take away visual distracters, such as the movements and gestures of the classroom participants, and thus they enable a focus on the spoken word. The transcription facilitated my noting of the use of particular words, phrases and patterns of speech. For example, each time a teacher talked about a "group of one" I could note the child or children's responses and gradual development of understanding. Patterns of speech were used to help step a child from a know fact to a challenge, such as "A week is seven days—and a fortnight?". Patterns of interaction were used for more subtle teaching purposes were also evident, such as each time when one teacher needed a correct answer so that she could use the child to help explain how it was obtained she called on one particular student.

The transcriptions are proving very useful in my discussions with the teachers. In practical terms, they have enabled me to give the teachers a section of the lesson well before an interview, so that they have time to look at and think about it—perhaps in the light of a question that I have also given them, or perhaps just as a section that might lead to a productive conversation.

They have also proved an easily navigable and searchable resources for re-visiting sections of the lesson. I had assumed that videotapes would be used for "simulated recall" and some of that process has been used, but organising to use transcripts has been much simpler, because the video room does not have to be booked to show a representation of a lesson snippet (perhaps a ten-second incident). Similarly, a contained section of a transcript (perhaps only a few sentences) can be handed to a teacher and become the focus of conversation, without distractions of preceding and subsequent interactions. Because the lesson transcript was in print format, it could more easily be presented alongside data drawn from other sources. For example, I was able to pull out all of the interactions a teacher had with a child to do with the development of a particular maths concept mathematics across a few lessons, and print them on the same page as a relevant section from an interview with the teacher.

However, some of the very factors that make oral language a rich data source were stripped out in the transcription process, including gesture, facial expression, bodily presence, and pauses (including the pedagogical tool of "wait time"). Similarly, where a child sat in a group, and the physical interactions between children (such as two children quietly talking together while the teacher was explaining a concept) were also no longer part of the available data. Cazden (1988) and Gee (1990) note that analysis processes much take account of the fact that texts are part of social situations that comprise of more than language. I found that the more contained a data form, the more difficult it proved to posit as the meaning of a text and to claim adequate evidence for any particular conclusion.

Photographs

My use of photographs within the thesis has several aims. The main one was to re-capture some of the raw data that had been stripped out, as noted above, and to do this in a form that could be presented in the body and the appendices of the printed thesis. While a short quotation from the text can convey some meaning, a photograph can enrich this remarkably. For instance, when a transcript line reads "It's thirteen!", aspects such as the expression on a child's face of whether (s)he has a hand raised may be important features of the dialogue that have been lost. Thus the use of still frames from the video allowed some representation, in a printed record, of the physical activity that was part of the classroom activity.

Like the transcripts, the photographs allowed selection from the complexity of the interactions and this enabled me to focus on particular incidents and small aspects of the interaction in my later deliberations. With the use of cropping, aspects of the frame that seemed particularly relevant to the topic at hand could be foregrounded. As Erickson and Wilson (1982) note, "Having one's visual ... attention directed one way or another is an intrinsic part of the experience of viewing and editing documents" (p. 6). Also as with the transcripts, photographs allowed time for deliberation: an image that stays in view for as long as it takes to talk about an aspect of it facilitates interpretation of the image by teachers and researchers as well as interpretation by the researcher of the meanings the teacher holds. The time afforded gave me opportunities to reduce dependence on premature and audio-dominated interpretations.

Still photographs also allowed for events in the background to be noted. In one frame, I noticed a child taking her shoe off during a game of Buzz. The teacher wanted the group to learn to count by twos, but when the child was put out of the game (that is, by chance she was number ten) she had "tuned out". Even though I had watched the video many times, I had focused only on the engaged pupils and the strategies that the teacher was using to build their understanding. Noticing the background information in that photograph took me back to the videos of the Buzz game and other activities where the same child was filmed to see whether she was ever engaged when not centrally involved in the discourse, and to compare her patterns of engagement with those of other children who did seem to develop the desired mathematical understandings and skills.

As Henry (1991) noted, when teachers watch a video, they tend to get "caught up" in the complexities of classroom interaction. Walker and Alderman (1972) claim that "the continuously moving image (on film of videotape) encourages the audience to identify with the actions (while) stop-frame ... sequences retain more strongly their identity as visual representations" (p. 136). I found that reflections on their own looks, clothing, expressions, vocal tenor, etc., or on aspects of the videoed interaction such as the activities of a wayward child, easily distracted teachers' attention from the focus of the interview, and that this was less likely to happen when a series of photographs was used.

As a researcher, using photographs (and later playing or fast-forwarding the tape with the sound turned off) took my attention away from vocal interactions enough to consider other aspects. Shifts in physical arrangements for participation signalled subtle shifts in social participation (termed *f-formations* by Kendon, 1990). "ownership" of a new idea, for instance, was very different in group and individual situations. Conversely, teachers' expectations for different patterns of social engagement led to children arranging themselves physically. When, for example, a teacher said, "I want you to record your own ideas and solution", children moved apart and used their bodies to shield their work, taking up a very different physical formation (and using a different idea of what it is to do mathematics) that they did when the teacher had indicated expectations for group problem solving.

However, photographs do not facilitate the study of connected sequences of physical actions, give a sense of time, link the visual with the aural, or cater for the development or cause-effect theories. Why does a child put out twelve counters as a group of ten and two ones, as requested, and then sit on them? Why does a child spend ten minutes drawing a clock when a quick sketch would have enabled him to work on the mathematics of the problem? Why does the teacher smile and nod (as if giving positive feedback) as she corrects the child's misconception? How is it that after repeated explanations, the child does still not seem to understand the notion of rotational symmetry? What is the meaning a boy's head-nodding as he watches a teacher's demonstration, then continues to use a completely different algorithm—and what effects did his nods have on the teacher's shaping of the explanation?

In comparison with videotape, photographs also suffer from the same lack of wider context as transcripts. The audience responds to the information it has by trying to fill in missing details in order to make sense of the data. Just as we create viable stories from comic strips and works of art, we interpret data by trying to make it fit with a known social system. The more data we have, and the more varied that data, the more we are likely to be able to create a story that creates a viable fit with the original event.

Ethical and Subjectivity Issues

With transcripts, it can be relatively easy to hide the identity of teachers, children and schools, although some useful critiques have been written on this position (see, for instance, Richardson, 1973). With photographs and videos, however, the task is near impossible. Some techniques are available: I have spent many hours blurring the figure of one child whose parents did not give permission for her to be filmed, but even this action draws attention to her unseen presence more than the use of a pseudonym in the relevant transcripts does. Of course most teachers, children and parents willingly give permission, but the point raised at MERGA in 1996 is an important one: What happens if a naive ten-year-old later becomes an important person and the video is then used for unethical purposes. This is not likely to be a problem if raw video data are used only in the analysis stage of research projects, but when they are made available within the thesis and its appendices, these are publicly available documents.

It has been a particularly difficult aspect of the project to write up events in ways that do not leave the teachers open to unfair comparison and criticism. Films of two teachers teaching the "same" lesson content inevitable set up a comparative situation, even if no comments are written to this end. Similarly, snippets of teachers who ignore a child for a long time or appear to be particular demanding of a child (perhaps for a legitimate reasons that observers cannot access), and excerpts of teachers giving muddled explanations that are always easy for an observer to replace with a more coherent statement, are further examples of the types of video data that lay teachers open to criticism. While the inclusion of such problematic material can be the subject of negotiation between researchers and their subjects, it is difficult to have such discussions without the participants' values—related to being a teacher (with public perception, promotion and personal-professional pride at stake) and being a researcher (with useful data at stake), as well as the traditional teacher-researcher hierarchy, presenting barriers to true negotiation.

Just as conversion to transcripts or photographs does not overcome ethical and moral problems that arise when videotapes are used to research mathematics classrooms, it does not assist one to make valid judgements as a researcher. Data gathering and presentation in all three visual forms outlines above raise problems to do with subjectivity. The first is inherent in conscious and unconscious decision-making about what is to be captured on video: camera operators attend to what captures their interest. While I started out with every intention of following a teacher, because I was interested in the roles that teachers play in developing children's mathematical understandings, this decision in itself posed dilemmas. For instance, there was much that happened as a consequence of an interaction with a teacher that I felt needed to be captured. For example, what a child wrote after an interaction with the teacher might demonstrate a growth of understanding arising from that interaction. But a focus on the child's writing would take the lens off the teacher, despite the fact that the next teaching event was in progress. As noted by Erickson (1991), although film records are relatively complete documents, they can never be complete, and decisions about what to record and how to record it are not neutral.

Also, paradoxically, what teachers do not do is as much a part of the teacher's activity in a classroom as what the teacher does. A child who obviously does not understand the mathematics problem that is being worked on and who is ignored for a lengthy time may be considered to be an important.component of the interaction, and a vital aspect of the teachers' work that needs to be recorded. When one is using a video camera to follow a teacher, decisions then have to be made about how to gather evidence of incidents such as this.

The selection of classroom events for representation, analysis and writing up is a value-laden exercise. I have been aware of this when making judgements about which lessons seemed most productive for explorations related to the project's research question, given that only limited space (and bytes) will be available in the final products. Is it better to search for concise snippets or to use longer ones that illustrate more clearly the point being made? Similarly, I have been conscious that selection of certain sections of lessons, and then particular snippets to transcribe has been a subjective process—and have found that talking and then writing about his proved more cathartic that helpful.

The most obviously uncomfortable research act in relation to the three visual formats for data presentation was the selection of video frames to convert to photos. For instance, in the time that it takes to say "What do you mean by 'equals'?", one frame shows a teacher leaning towards a child and looking interested, while in another she is backing off and looking into what appears to be the distance while appearing to be quite disinterested. The transcript implies interest, and the video shows that she has been temporarily distracted by another child but immediately re-engages in the conversation. However, a single photograph indicates neither. A series of photographs would, but then that virtually re-forms that section of the video.

A similar situation is faced when one has to decide how to frame a photograph. It makes sense to keep a video shot as wide as possible, because "the concern will be with the limits and boundaries of attention rather than with their forms" (Walker & Alderman, 1975). When writing up a project and selecting appropriate photographs, it makes sense to frame the action so that it brings pertinent areas of the picture closer. This situation is little different from the selection of any data to support arguments being made by researchers. (Which incidents?, Which sentences?) However, I was more conscious of the potential effects when choosing parts of a photograph to be included or cropped. The enforced choice of frame, together with later choices about how much of the frame to include, raises questions about what story (in a post-structuralist sense) I, as a researcher, want to have represented by the photograph. Perhaps I wish to convey the message that continual interruptions in busy classrooms make it impossible to give enough attention to each child's cognitive and verbal functions, or that teachers value classroom control above the development of individuals' understandings, or that expert teachers can use a quick gaze with subtle characteristics such as raised eyebrows or pursed lips without interrupting a productive mathematical dialogue. It is not a matter of my presenting some objective truth—there would probably be elements of each of the above in this incident, but in any case I have no way of knowing their relative influences. There would also be other stories that I have not even considered, or stories (such as genuine disinterest by the teacher) that I may have thought about and dismissed because of my other experiences in the case study site. It is also not a matter of asking the teacher what she was thinking at this time, for we are not aware of many of the movements we make as professionals, let alone the reasons for them; and even if she were aware, the response would be likely to be affected by the research context.

Selection of any data is subjective, and the greater the reduction (the more that the result is decontextualised over area and time), the greater is the potential for biased choice. What needs to be recognised is that with any medium the selection processes themselves must be the subject of discussion. In relation to the use of photographs as research data, Walker (1985) notes that this can bring strength to the project:

> Photographs, ... long unused in observational research because of the problems of subjectivity and bias associated with the taking and processing of pictures, provides a more acceptable stimulus once selectivity becomes part of the subject of the research. ... A slight shift in angle, the inclusion of exclusion of contextual information, the choice of one split-second rather than another provide similar pictures open to very different interpretations. This weakness of the photograph as data can be turned to a strength when it is used to elicit responses or to communicate complex messages. ... part of the task may be to replay events to the people involved to get then thinking and talking reflectively ... What is important about the picture is determined, in part at least, by what people say about it. (p. 143)

Conclusion

Videotapes on mathematics classrooms, however used, provide limited data. Ethnographic research involves analysis of particular recorded events "in relation to the broader circumstances of choice and constraint within which the event occurs" (Erickson, 1992) so microanalysis of audio-video needs to occur in the light of other data about the broader personal, institutional and socio-cultural circumstances, networks and discourses of the research context. The research methods that I have outlined above do not provide this broader context—but they have assisted with the data analysis process.

Recognising its limitations in ethnographic research, in comparison with other tools, what videotape does capture is relatively complete. A wealth of information is available for description and interpretation. We can use this resource as it is, we can link it to other available data, or we can enrich its use by employing its various aural and video components. I have attempted to outline the advantages of using three individual visual products derived from the filming process, suggesting that they allow deliberative and focused attention on various aspects of the classroom interactions. However, I recognise that such research processes are not unproblematic, and that the resulting formats do not overcome many of the dangers inherent in the use of videotapes in mathematics education research.

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A research-based framework for assessing early multiplication and division strategies

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This paper describes the integration of multiplication and division strategies within a research-based Learning Framework in Number (NSW Department of Education and Training, 1998). The framework, consisting of six levels of multiplication and division knowledge is described in order of increasing sophistication of modelling, counting, grouping and sharing processes. Assessment tasks show a progression from initial grouping and counting to abstract composite units, to repeated addition and subtraction and to multiplication and division as operations. The framework, which is integral to the Count Me In Too Project is currently being implemented in years K-2 in NSW schools.

The purpose of this paper is to describe how long term research on children's development of multiplication and division has been translated into a Learning Framework in Number in order to assist professionals in assessing and improving children's number knowledge. The paper describes the theoretical framework and research basis which includes longitudinal research of 4-9 year olds development of multiplication and division though clinical interviews and teaching experiments. The main focus of the paper is to describe explicitly, the links between levels of development and key assessment tasks so as to distinguish how we can promote the development of increasingly sophisticated multiplicative strategies in young children.

Background to the Framework

Recent studies show that young children can develop multiplication and division concepts in the first years of schooling and this highlights a problem that teaching practices are not necessarily focussed on children's potential mathematical development (Carpenter, Ansell, Franke, Fennema, & Wiesbeck, 1993; Clark & Kamii, 1995, Kouba, 1989; Hunting & Davis, 1991; Mulligan & Mitchelmore, 1997). There is also growing evidence that once children reach the primary grades they are unable to solve problems involving multiplication and division or apply multiplicative number facts with meaning. In the upper grades, students find difficulty in using multiplicative reasoning in a range of contexts and integrating their understanding of rational number with multiplication and division (Behr, Harel, Post, & Lesh, 1994; Bell, Greer, Grimison & Mangan, 1989; Confrey & Smith, 1995).

Multiplicative reasoning is essential in the development of concepts and processes such as ratio and proportion, area and volume, probability and data analysis. It is also clear that failure to develop multiplicative structures in the early years impedes the general mathematical development of students into the secondary school, for example, in using algebra, functions and graphs. A concomitant problem is that multiplicative concepts are often not well understood or well taught by teachers at primary and secondary level (Graeber, Tirosh & Glover, 1988; Simon, 1993). It appears that difficulties faced by older students can be attributed, at least in part, to the lack of development of an equal-grouping structure in early concept formation (Mulligan & Mitchelmore, 1997; Steffe, 1994).

An Overview of Research on Multiplication and Division

Studies investigating multiplication and division processes with younger children have identified the development of sound problem-solving strategies from an early age and the importance of modelling and representation in this development (Anghileri, 1989; Carpenter, Ansell, Franke, Fennema & Weisbeck, 1993; Clark & Kamii, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1997; Steffe, 1994). Recent research on multiplicative reasoning has looked to the early development of multiplication and division and fraction concepts, through essential processes such as counting, partitioning, grouping, unitizing and 'splitting' (Confrey, 1994; Hunting & Davis, 1991; Lamon, 1996; Steffe, 1994).

In longitudinal analyses of young children's intuitive models for multiplication and division problems Mulligan & Mitchelmore (1997) found that the intuitive model employed to solve a particular problem did not necessarily reflect any specific problem feature but rather the mathematical structure that the student was able to impose on it. Students acquired increasingly sophisticated strategies based on an equal-groups structure and calculation strategies that reflected this. Counting strategies were integrated into repeated addition and subtraction processes and then generalised as the binary operations of multiplication and division. Strategies used with concrete and sensory models were internalised and replicated at an abstract level with increasing sophistication. Although multiplicative reasoning does not usually emerge in instructional programs until the second or third grade the Count Me In Too Project assesses the development of these processes in an attempt to seek out new ways of addressing the difficulties experienced in learning and to formulate more valid assessment techniques than traditional multiple choice tests.

Classroom-based studies on teaching and learning multiplication and division have also investigated the ways in which children devise and represent related problem situations and solve computations (Boero, Ferrari & Ferrero; Carpenter et al. 1993; Lampert, 1990; Mulligan & Mitchelmore, 1996; Murray, Olivier & Human, 1992; Nesher,1988). Teaching approaches based on the the Count Me In Too Project aim to encourage closer links between concrete and abstract thinking in order to promote increasingly sophisticated strategies based on the levels of described later in Table 2.

The development of composite structure; A developmental framework assessing the growth of multiplication and division processes must be based on the acquisition of an equal-grouping (composite) structure which is at the heart of multiplicative reasoning. A composite whole is a collection or group of individual items that must be viewed as one thing. For example, a child must view three items as "one three" in order for the unit "three" to be a countable unit. For a true understanding of multiplication and division the child needs eventually to co-ordinate groups of equal-sized groups and recognise the overall pattern i.e. composites of composites, e.g."three sixes". Steffe (1994) describes this as a premultiplying scheme, "For a situation to be established as multiplicative, it is necessary at least to co-ordinate two composite units in such a way that one of the composite units is distributed over elements of the other composite unit" (p.19). Other theorists emphasise the importance of this structure calling it unitizing (Lamon, 1996), and re-initialising (Confrey, 1994). As well, multiplication and division may require that quantities or the numerical referents are changed or transformed as a result of the process. The quantity that is the product is a different type of quantity to the two like or unlike quantities that have been multiplied, e.g. where combinations of t-shirts and jeans produce "outfits".

Once the initial elements are developed and consolidated with repeated addition or repeated subtraction and sharing models, multiplicative reasoning must extend beyond these to a point where the commutativity of multiplication is recognised and the inverse relationship between multiplication and division is applied. The development of multiplication and division as inverse processes forms the basis of a developmental model of composite structure. The acquisition of multiplication and division as binary operations relies on the child's ability not only to develop composite structure and commutativity but also to recognise the relationship m x n where m is the composite unit "operated upon" n times. This is quite different to a repeated addition notion of multiplication which is commonly used in teaching practice.

Children may use identical or similar strategies for solving both multiplication and division tasks except that in division, the child will form and count composite units from an existing quantity. Interestingly, it has been found that division is not necessarily more difficult than multiplication, and in many situations, division situations may be easier than multiplication. For example, it may be easier for a child to share counters into equal groups and count the number of groups rather then keeping track of a larger number of composite groups for multiplication. Teaching children to share and group small numbers into equal parts can facilitate the development of multiplication and division strategies ie. non-count by ones strategies (Mulligan & Mitchelmore, 1996).

The Learning Framework in Number

Five key components shown in Table 1 have been developed as key aspects of early number learning. Together, parts A, B, C, D, and E form an interrelated framework (see Wright, 1998). While early arithmetical strategies form the central structure of the framework other aspects such as multiplication and division should not be seen as compartmentalised or in developed in isolation.

Children's early multiplication and division knowledge is based fundamentally, on the development of counting sequences and arithmetical strategies (Part A), along with skills of combining, partitioning and patterning (Part E). The concepts of multiplication and division are not only interrelated, but are closely linked with addition and subtraction strategies, and early fraction learning. For a true understanding of multiplication and division the child needs to eventually co-ordinate groups of equal groups and recognise the overall pattern ie. composites of composites eg. "three sixes".

Table 1: Aspects of Early Number Knowledge

A1	*Early Arithmetical Strategies*
0	Emergent Counting
1	Perceptual Counting
2	Figurative Counting
3	Counting On
4	Facile Number Sequence
B1	*Forward Number Word Sequence*
B2	Backward Number Word Sequence
B3	Numeral Identification
C1	*Early Multiplication and Division Strategies*
1	Initial Grouping and Perceptual Counting
2	Intermediate Grouping and Counting
3	Abstract Composite Units
4	Repeated Addition and Subtraction
5	Multiplication and Division as Operations
C2	*Fraction Knowledge*
D	*Base-Ten Arithmetical Strategies and using "Five"*
D1	Base-ten Strategies
D2	Quinary-based Strategies
E	*Early Arithmetical Procedures*
E1	Combining and Partitioning
E2	Spatial Patterns and Subitising
E3	Temporal Sequences
E4	Finger Patterns

Table 1 places multiplication and division strategies in the central section of the framework. It should be noted that children who have progressed only to Figurative Counting (A1.2) in early arithmetical strategies are unlikely to construct composite units as counting units (iterable units). However, children who are able to partition using five and ten as a base, including quinary-based D1 - D2, make "five", partitioning "five", and counting in fives using equal groups, are likely to be developing composite units.

The development of combining, partitioning and patterning strategies included in Part E as well as other early arithmetical strategies of addition and subtraction can be effectively linked with the development of multiplication and division strategies. Various aspects of multiplication and division can be seen as integrated or overlapping with these strategies. Thus, two or more different situations involving multiplication and division may elicit the same arithmetical strategies.

The Development of Multiplication and Division Strategies

A framework consisting of six levels of multiplication and division knowledge is described in order of increasing sophistication of modelling, counting, grouping and sharing processes. At the same time, the development of multiplication and division as operations relies on the child's ability to co-ordinate composites ie. use groups of equal groups as single entities. Levels of this framework are based on longitudinal analyses of children's development of multiplicative strategies from Grade 2 through 3 (see Mulligan & Mitchelmore, 1997).

Six levels are described below in Table 2 giving descriptions of the main types of strategies. It is noted that children may demonstrate a combination of these strategies depending on the size and type of the numbers used. For example, children find counting and grouping in 2's, 5's and 10's easier than 3's or 4's. Generally, children will progress from *initial composite counting*, to *abstract composites*, to *multiplication and division operations*, and to *known facts.*

Table 2 Development of Multiplication and Division Strategies

C1.1 Initial Grouping and Perceptual Counting: models or shares by dealing in equal groups but they do not see the groups as composite units; count each item by ones *(perceptual).*

C1.2 Intermediate Composite Units: models equal groups and counts using rhythmic, skip or double counting; counts by ones the number of equal groups and the number of items in each group at the same time only if the items are visible.

C1.3. Abstract Composite Units: models and counts without visible items ie the child can calculate composites when they are screened, where they no longer rely on counting by ones. The child may not see the overall pattern of composites such as "3, 4 times".

C1.4 Repeated Addition and Repeated Subtraction:: co-ordinates composite units in repeated addition and subtraction. Uses a composite unit a specific number of times as a unit e.g. 3 + 3+ 3 +3; may not fully co-ordinate two composite units.

C1.5 Multiplication and Division as Operations:
Two composite units are co-ordinated abstractly eg. "3 groups of 4 makes 12"; "3 by 4" as an array; "3...4 times as many, "12 into 3 groups...that's four; "3, four times".

C1.6 Known Multiplication and Division Fact Strategies: recalls or derives easily, known multiplication and division facts; uses multiplication and division as an inverse relationship.

Assessment tasks; Several critical assessment tasks are described explicitly in Appendix A. These tasks form a basis for ascertaining the child's level of development of composite structure. Appendix A shows a progression from counting strategies based on multiples to the formation of composites without visible models. As for other arithmetical tasks (Table 1, A1), the assessment focusses on the child's ability to use strategies that are increasingly more sophisticated and assesses the child's ability to use composite units abstractly, i.e. calculate groups and the number of groups when items are screened. For multiplication and divison tasks there is a distinct difference in difficulty between partition and quotition tasks (Appendix A , C4.2 and C4.3) and the importance of an abstract double counting strategy (C4.4) when items are screened.

Discussion

The development of efficient counting (non-count by ones, skip and double counting) and composite units are integral to developing composite structure. Co-ordinating composite units, e.g. " three threes as a unit of 9" depends on the ability to move beyond counting based on a unitary notion and to use a pattern of multiples as a double count ("1, 2, 3 (one), 4, 5, 6 (two)" etc.) mentally. While the development of direct counting and visual modelling precedes development of abstract composite structure there exists a complex interrelationship between counting and composite structure at the abstract level. The use of skip and double counting procedures gives rise to more efficient processes that take advantage of the equal-grouping structure where repeated addition (or subtraction) is generalised as an operation.

The development of repeated addition or repeated subtraction at level C1.4 (see Table 2) does not constitute a full conceptual understanding of multiplication or division. The next level C1.5 distinguishes the development of multiplication and the related division process as the distribution of a composite unit across elements of another composite unit e.g. generalising the structure of composites, for example, as "six, three times as 6 x 3 = 18". Critical to developing this relational understanding of multiplication is the ability to see multiplication and division in an inverse relationship and to explain commutativity such as 6 x 9 = 9 x 6. Children who are able to simply recall multiplication and division number facts without being able to explain and represent the composite structure are not yet functioning at this level.

Implications

The assessment and development of multiplication and division strategies in the early years of schooling requires professionals to integrate key aspects of developing composite structure within number learning generally. Although grouping and sharing processes may on the surface, form part of "traditional" practice, the systematic and explicit nature of the framework allows professional to gain further insight into matching learning experiences with the child's potentialities. Anecdotal evidence of the initial implementation phase has indicated that children in years K-2 are capable of using multiplication and division strategies effectively across a range of learning situations. The implementation and extension of the framework into more complex problem-solving tasks will have implications for curriculum reform and teaching practices in the early years.

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Appendix A : Assessment Tasks for Early Multiplication and Division

C1.Tasks Assessing Number Word Sequence in Multiples

The purpose of these tasks is to assess the child's facility with the forward number word sequence in multiples of 2 and 3, i.e. skip counting by twos and threes. This is a basic task where the teacher can discriminate between the child's facility with skip counting produced verbally, by co-ordinating skip counting with visible items, and by co-ordinating skip counting with a numeral track.

C1.1 Counting forward number word sequence in multiples of 2 and 3

The teacher asks the child to begin counting in twos until directed to stop beginning with the number 2. When the child has reached 20 the teacher directs them to stop. The task is repeated with directions to count in threes but it is anticipated that this is more difficult. When the child has reached 12 the teacher directs them to stop.

C1.2 Counting visible items in a row by multiples of 2 and 3

The child is presented with a row of 20 counters and is directed to count them in twos. The teacher observes whether the child co-ordinates correctly the number word sequence of twos with the items. The task is repeated with directions to count a row of 12 counters in threes.

C1.3 Counting by multiples of 2 and 3 using a numeral track

The child is presented with a numeral track 1-10 and is directed to count forwards in twos by correctly co-ordinating (by touching) the numerals with a verbal count. The task is repeated for counting by threes. An extension of this task is to ascertain whether the child can identify hidden numerals in a sequence of twos or threes. Several numerals in multiple patterns of twos or threes are covered. The child has to identify the hidden numerals.

C2 Tasks Assessing Equal Grouping and Sharing with Visible Items

The purpose of these tasks is to establish whether the child can construct **initial composite** units by equal grouping in multiplication and division situations. There are three basic tasks where the child makes specified groups by grouping and sharing. The teacher observes whether the child is unable to coordinate the number of equal groups and the number in each group at the same time.

C2.1 Composite units task by equal grouping (multiplication) using visible items

The child is asked to make specified equal-sized groups e.g. make 4 groups with 3 in each group. This is done using one to many matching where the number of groups is already modelled. The child places 3 lego people on each lego train or 3 counters on each plate where the lego train or the plate represents the number of groups. The child is asked to calculate the total number of items.

C2.2 Composite units task by sharing (partition division) using visible items

The child is asked to share a specified number of items into a specified number of groups and calculate the number of items in each group e.g. share (equally) 12 items among 3 children, how many do they have each? This is done using one to many matching where the number of groups is already modelled e.g. items to represent children.

C2.3 Composite units task by grouping (quotition division) using visible items

The child is asked to make specified equal-sized groups where the number of groups is unknown and to calculate the number of groups e.g. 12 items are shared among the children and they get 4 items each. How many children are there? The teacher observes how the child forms groups of 4 items until the 12 items are exhausted.

C3 Tasks Assessing Use of Composite Units Involving Partially Screened Items

The purpose of these tasks is to establish whether the child can construct **intermediate composite** units by equal grouping in multiplication and division situations. The basic tasks (C2), where the child makes specified groups by grouping and sharing using visible items and uses referents for the groups, are repeated but this time the items are partially screened. The teacher observes whether the child is unable to coordinate the number of equal groups and the number in each group at the same time.

C3.1 Composite units task by equal grouping (multiplication) partially screened

The child is asked to make specified equal-sized groups e.g. there are 4 containers with 3 items in each container. e.g. The child places 3 items in each of 4 containers (opaque). The child is asked to calculate the total number of items in the containers where the items cannot be seen or counted.

C3.2 Composite units task by sharing (partition division) partially screened

The child is asked to share a specified number of items into a specified number of groups and calculate the number of items in each container e.g. share (equally) 12 items into each of 3 containers, how many items in each container? The child is required to keep track of the number of items in each container where the items cannot be see or recounted.

C3.3 Composite units task by grouping(quotition division) partially screened

The child is asked to make specified equal-sized groups where the number of groups is unknown and to calculate number of groups e.g. 12 items are shared into containers where the child knows there are four in each container. The teacher observes how the child forms groups of 4 items until the 12 items are exhausted. The child is required to keep track of the number of counters in each container where the items cannot be see or recounted.

C4 Tasks Assessing Use of Abstract Composite Units

The purpose of these tasks is to establish whether the child can construct **abstract composite** units by equal grouping in multiplication and division situations. The basic tasks (C2) are repeated without any concrete or visible items. The child is required to coordinate the number of equal groups and the number in each group at the same time. An abstract double counting task (C4.4) naturally evolves from the C4.3 task with grouping (quotition division) and is important for assessing coordination of composite units.

C4.1 Composite units task by equal grouping (multiplication)

The child is asked to calculate the total number of items in an equal grouping problem where the items cannot be seen or counted in an equal grouping problem e.g. there are 4 groups with 3 items in each group. How many are there altogether?

C4.2 Composite units task by sharing (partition division)

The child is asked to calculate the number of items in each group e.g. share (equally) 12 items into 3 groups, how many in each group? The child is required to keep track, mentally, of the number of items in each group where the items cannot be see or counted.

C4.3 Composite units task by grouping (quotition division)

The child is asked and to calculate the number of groups where the number of groups is unknown e.g. 12 items are shared into groups where the child is told there are four items in each group. The child is required to keep track of the number of items in each group where the items cannot be see or counted

C4.4 Abstract double counting task by grouping (quotition division)

Repeat task C4.3. or ask similar division tasks e.g. there are 12 biscuits and the children are given 3 biscuits each. How many children are there? The teacher observes whether the child coordinates the number of groups and the number of items in each group simultaneously by asking the child to talk aloud as they solve the task. The teacher observes whether the child uses a referent such as fingers to keep track of the number of groups e.g. " 1, 2 , 3 (1)...4, 5, 6, (2), ...7, 8, 9,(3)... 10, 11, 12 (4). The task can be repeated using more difficult numerical grouping such as 5 groups of 3.

C5. Tasks Assessing Use of Composite Units in Arrays

Co-ordinating the equal-sized groups (composite units) as rows and columns and recognising the commutativity of multiplication e.g. 3 x 5 = 5 x 3, is an essential aspect of developing multiplication and division concepts.

C5.1 Calculate total number of items in a partially screened array

This task assesses the use of intermediate composite units where the visible composite units are used to count those that are screened. The child is presented with a 3 x 5 rectangular array of items with two rows screened. The child is told that there are two rows exactly the same as the other rows and asked to calculate the total number of items.

C5.2 Recognise commutativity of multiplication in an array

This task assesses the commutativity of multiplication where the composite units are visible. Present a rectangular array of 3 x 5 items fixed on a card e.g. stickers in rows and columns. Turn the array at right angles. Ask the child whether there are still the same number of items in total and to explain the grouping.

C5.3 Calculate total number of items in a screened array

This task assesses the use of **abstract composite units** where equal groups are visualised and counted when completely screened. A rectangular array of 3 x 5 items is screened. The child is required to visualise the screened array from a description of the number of rows and the number of items in each row and calculate the total number of items.

A graphical approach to the teaching of introductory calculus

Bruce Nevill, John Wycliffe Christian School, Blaxland

This research project investigated the use of computer software (ANUgraph) in the development of calculus concepts. One group of students completed the calculus unit following a traditional teacher-exposition method. The other group covered the same content using a graphical approach, which relied heavily upon the use of the computer software to introduce and reinforce key concepts. Results indicate that the lessons proved successful in increasing students' understanding of the fundamental concepts of calculus.

Rationale and Theoretical Framework

A traditional method of teaching Differential Calculus is to introduce students to a formula which includes a limit. A typical lesson sequence often involves students completing "first principles" exercises before learning a rule for differentiation and this sometimes leads to a separation of the process of differentiation from the concept of the gradient function or derivative. Tall & Vinner (1981) identified the concept image / concept definition conflicts that occur in students when using the traditional approach and student's difficulties with the algebraic manipulations involved in the "first principles" definition were described by White (1992) and Arnold (1994). White (1991) developed a new approach to the introduction of calculus concepts which attempts to avoid the conceptual difficulties described above and uses practical contexts in which to investigate the nature of change. This work coupled with the developments that have occurred in the technological field as described by Tall (1986) and in particular the widespread availability of "Mathematical Function Plotters" have lead to the development of the approach used in this study.

The emphasis, in this study, is on the use of the computer to model the concepts being introduced from a graphical perspective. Thus the lesson contexts are generally graphical rather than practical, in the sense used by White (1992). Particularly significant, is the use of the graphical approach developed by Tall (1986) to introduce the concept of the derivative using the "zoom" function available in the software. This allows for the conceptual problems associated with limits and function notation to be circumvented thus enabling the students to gain a clear concept definition of the gradient of a function prior to having to engage in the algebraic manipulation associated with the derivative. Also of particular importance here is the approach taken to develop the elementary rules of differentiation from the graphs of functions rather than from the "first principles definition".

The Study

Methodology

A quasi-experimental design was used to test the effects of the use of a Mathematical Function Plotter (ANUgraph) and associated pedagogical approaches in the teaching of Elementary Calculus to a Year 11 - 2 Unit Mathematics Class. A Trial group was used to assist in the development and testing of pre-test and post-test materials and lesson plans. The selected treatment group (a class of 16 students) and the control group (a class of 23 students) were tested using the pre-test and it was found that there was no significant statistical difference between the groups on the overall test results. [t (39) = 0.618 with p=0.470, (> 0.05)]. The treatment using the graphical computer-based approach was applied and the control group was taught using a traditional "limit based first principles" approach. Both groups were tested using the Post Test. Results revealed a 12% difference in overall means, (p= 0.035; $p < 0.05$ $n=39$)

Treatment

Having developed and fine tuned all materials the Pre-Test was administered to both the treatment and control groups. This was followed by the lesson sequence and Post-Tests. The responses of students in *pre-tests* and *post-tests* were used to ascertain their perceptions and understanding of key concepts associated with introductory calculus. These concepts included the nature and definition of function, the definition of derivative, the use of derivative in relation to properties of functions, the use of second derivative as a tool in investigating the nature of functions.

Teaching Strategies

Example 1: Comparative Sequence of Stages in the Teaching Process

Stage	Treatment Group	Control Group
1	Review of Functions & Function Notation	Introduction to Limits
2	Gradient of a Curve and the Gradient Function using ANUGraph Students Derive Formula : If $f(x) = x^n$ then $f'(x) = n.x^n$	Complete algebraic exercises on limits.
3	Differentiation of Sums and Differences	Introduction to gradient of a curve & derivation & use of the formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ Exercises on the use of formula
4	Derivative as a Rate of Change and Increasing/Decreasing Functions.	Rules for Differentiation Formula :If $f(x) = x^n$ then $f'(x) = n.x^n$ *Conditions for Differentiability*
5	Definition of the Derivative as a Limit	Differentiation of Sums and Differences
6	Curves that Have Some Interesting Points *(Conditions for Differentiability)*	The Chain Rule- Exercises
7	The Chain Rule	Further Exercises on Chain Rule
8	The Product Rule/Quotient Rule	The Product Rule and Quotient Rule- Exercises
9	Consolidation and Practice	Exercises on differentiation
10	The First Derivative and Curve Sketching	Turning Points and Local Maxima and Minima- Exercises
11	The Second Derivative and Curve Sketching	The Second derivative and curve sketching - Exercises
12	Further Applications , consolidation and practice.	Consolidation and practice.

Major Findings

Comparisons Between Groups

Pre-Test Results: Initially the mean and standard deviation for each key concept category was calculated for the Pre-Test providing an overall summary of the results. In addition to this, the results were analysed to ascertain if any significant statistical difference occurred between the performance of each group on each concept/skill area as well as the overall test result. An unpaired two-tailed t - test was used for this purpose and the results for each question and the total test score are include in the Table below.

Group A - Treatment Group; Group B – Control Group; t -values are unpaired 2-tailed, with 39 df and level of significance at 95%. Scores are percentages

	Group	Mean	SD	t(39)	p	Comments
TOTAL	A	52	15	0.73	0.470	No sig. diff. on overall total.
SCORES	B	49	17			

The results indicate that the overall result of t (39) = 0.618 with p=0.470, (> 0.05) supports the null hypothesis, leading to the conclusion that there is no significant difference between the groups on the overall scores.

Post-Test Data

Post-Test results for the Treatment and control groups were also analysed in a similar manner. The tables below show the results for each of the groups on each question.

	Group	Mean	SD	$t(39)$	p	Comments
TOTAL	A	59	20	2.187	0.035	A significant difference occurs on the overall test results. The difference of 12% between means is statistically sig,. @ 95% confidence limits. $t(39) = 2.178$ with $p = 0.035$, (<0.05.)
SCORES	B	47	19			

Conclusions & Implications

The results obtained demonstrate that the students in the treatment group were able to successfully complete the standard algorithms of differentiation and interpret these results in a superior manner to those in the control group. In comparing the two approaches taken here one of the elementary differences is that the treatment group spent more time on "playing" with the graphs of the functions and less time on practice exercises in the traditional sense. It would appear therefore that the students' competence in the algorithms of differentiation did not suffer, but was in fact higher, and was therefore enhanced by this approach. Perhaps the reason for this was related to the fact that the graphical approach places the concepts in a context thus enhancing the students links between concept image and concept definition. The use of the "magnification" principle in determining the gradient at a point and its initial link to the rate of change at that point, avoids the complications of the limiting process and associated notations. The better performance of the treatment group on the standard algorithms of differentiation appears to be unrelated to prior understanding of the limiting process. The fact that there was little difference in the initial understanding of the limiting process as measured in the pre-test substantiates this. The results in the post-test also supports the view that the graphical approach taken in this study enhances the concept image that the students have in relation to the derivative and its application.

The implications for teaching programs and curricula are that, (1) students should be given access to this type of technology in the context of the Mathematics curriculum in order to dynamically explore the graphical approach to the teaching of calculus; (2) the work supports earlier work by others in the development of a context-based approach to the teaching of calculus; and (3) a change in the emphasis in senior curricular which emphasise the graphical approach rather in addition to the more traditional algebraic limiting value approach is worthy of consideration.

Appendix 1.

A Sample Lesson Outline (Stage 2)

Introduction to the Concept of the Gradient Function (Stage 2)

The concept of speed being equivalent to the slope of a graph is developed from a known situation (ie. the Travel Graph).

The concept of average speed is developed and the link to the Chord between the two points on the curve is established.

The "zoom in" feature of the software is then used to investigate the curve in the neighbourhood of x = 1 to establish the gradient at this point.

This gradient is then linked to the concept of "Instantaneous Speed" which is then linked to the Gradient of the Tangent.

Later Exercises are used to develop the concept of the Gradient Function and hence the Derivative, using the "Differentiate Option" on ANUgraph to display a table of values of the Gradient Functions for various functions.

These values are plotted to produce graphs from which the students "guess" the function rules for the Gradient Function.

After several different repetitions of this process the rule for determining the derivative of a polynomial type function is derived from the pattern.

Stage 2 (Details of Teaching Strategies)

Student Investigation: (Teacher directed).

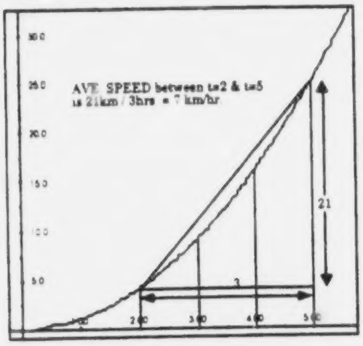
A vehicle travels from A to B and it's distance D kilometres from the starting point A is given by the function $D = t^2$; where t is the time taken in hours.

- How far is it from the origin after 5 hours?
- How long does it take to travel 16 km?
- What is its average speed over; a) the first 2 hours? (b) the first 3 hours? (c) the first 4 hours? (d) between $t = 2$ hrs and $t = 5$ hrs.
- What would it's speed be when $t = 1$ hour? Explain how you got your answer.

1. *Using ANUGraph*

Using ANUgraph, the students then plot the graph of D = t2
(Using F(x) for D and the x axis for t) (.ie. Graph $F(x) = x^2$)
Ensuring that the "*Show Coordinates*" option is selected (Special Menu), so that coordinate values on the graph can be identified using the mouse pointer, students investigate the answers to the preceding questions with reference to the graph.. Also it is pointed out that since we are investigating the position of the vehicle after it leaves its starting position we need only consider $t > 0$.

Speed: *The teacher now makes the following observations with the group;*



The average speed of the vehicle is given by

Speed = \f(Distance Travelled, Time Taken) .

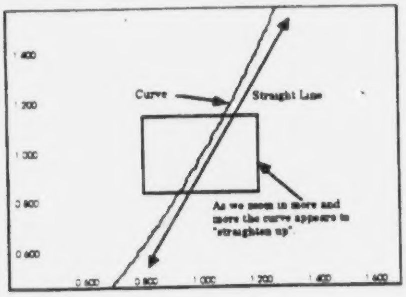
Average speed can be determined by joining two points on the curve and calculating

Ave. Speed = \f(Vertical Rise, Horizontal Run)

for the two points chosen. (See diagram). Therefore, on our graph *speed* is equivalent to the

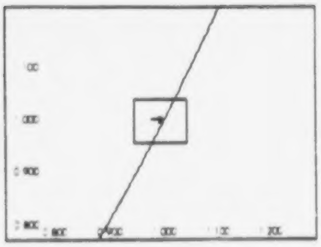
Slope = \f(Rise, Run) .

The teacher then proceeds to make the necessary connections to the instantaneous speed as follows;



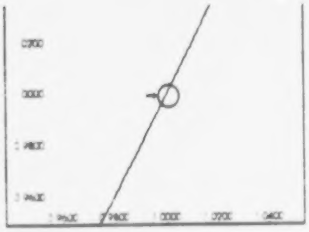
We can see therefore that the *Slope* or *Gradient* of the function can be used to determine the *speed*, however there is a problem here in that we are dealing with a curve rather than a straight line.

The instantaneous speed is a little more difficult to find but not impossible. Finding the speed when $t = 1$ hr requires an examination of the graph in the "neighbourhood" of $t = 1$.

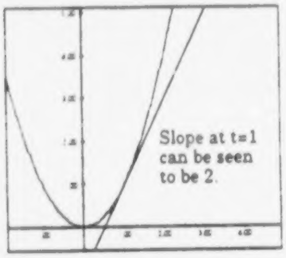


Using the "Zoom" feature on ANUgraph we can "zoom in" on the point (1, 1) and examine the slope to establish the speed of the vehicle at this point.

We notice that as the zoom area gets smaller the graph appears to get straighter.



We can therefore calculate using the formula - Slope = \f(Rise, Run) that the slope of the graph near $t = 1$ would be about 2. *(As we zoom in closer and closer we could draw a line with slope exactly 2, just touching the graph at (1, 1)*. (ie a tangent to the curve at t=1). In our problem this then gives us the "*Instantaneous Speed*" when $t = 1$ (ie. at time $t = 1$ hour, the vehicle is travelling at 2 km/hr.)



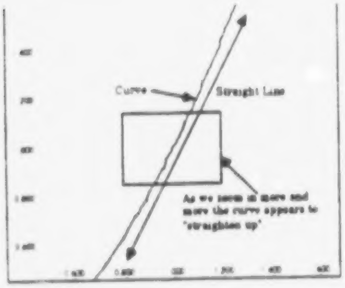
Slope at t=1 can be seen to be 2.

To assist us with the process of calculating instantaneous speed, ANUgraph will draw in the slope at a particular point by drawing the "*tangent*" at that point and calculating its slope. This option is available under the "Special Menu", it is called "Differentiate". *Differentiation is the process involved in finding the Gradient Function* of a given function.

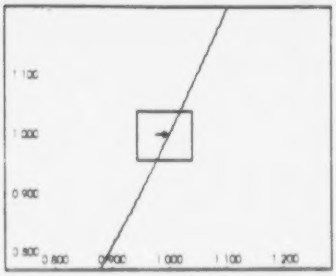
2. *Using the Sharp EL-9600*

Using the Sharp EL-9600 plot the graph of $D = t^2$. (Using Y -axis for D and the X-axis for t) (.ie. Graph $Y1 = x^2$). Press \x(Y1) \x(X/Q/T/N) \x(X^2) \x(ENTER) Press \x(GRAPH) to see the graph. Using the TRACE option, the coordinate values on the graph can be identified using the pointer, investigate the answers to the preceding questions with reference to the graph.. Also since we are investigating the position of the vehicle after it leaves its starting position we need only consider $t > 0$.
(Use the \x(WINDOW) button to adjust the Domain if you need to.)

Speed:

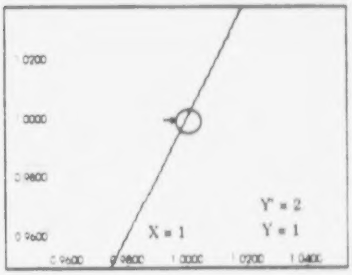


Instantaneous Speed; We can see that the *Slope* or *Gradient* of the curve(function) can be used to determine the *speed*, however there is a problem here in that we are dealing with a curve rather than a straight line. the instantaneous speed is a little more difficult to find but not impossible. Finding the speed when $t = 1$ hr requires an examination of the graph in the "neighbourhood" of $t = 1$.



By using the "Zoom" feature on the calculator we can "zoom in" \x(ZOOM) \x(3) on the point (1, 1) and examine the slope to establish the speed of the vehicle at this point.

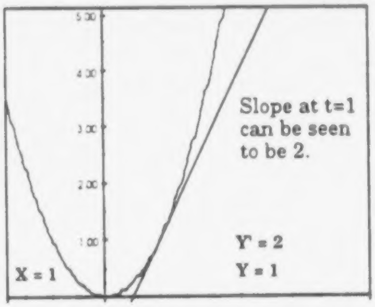
We notice that as the zoom area gets smaller the graph appears to get straighter.



(As we zoom in closer and closer we could draw a line with slope exactly 2, just touching the graph at (1, 1). (ie A tangent to the curve at t=1). The slope of this tangent would be the same as the slope of the curve at that point. To do this on the calculator use;

\x(2nd F), \x(DRAW), \x(5 T_LINE),\x(ENTER)

move the cursor to the point (1,1) and press \x(ENTER).



Slope at t=1 can be seen to be 2.

Zooming Out \x(ZOOM) \x(4) reveals that this line is a tangent to the curve and its gradient is the gradient of the curve at (1, 1). The Sharp EL-9600 allows us to calculate this value:\x(2nd F), \x(D Y'), \x(1 ON), \x(ENTER), \x(GRAPH), \x(TRACE)

Y' appears on the screen showing the value of the slope of the tangent at the designated point to be equal to 2.(ie. at time t = 1 hour, the vehicle is travelling at 2 km/hr.)

To assist us with the process of calculating instantaneous speed, the calculator will draw in the slope at a particular point by drawing the "tangent" at that point and calculating its slope.

We can DRAW a TANGENT at a point using the Calculator
Press \x(GRAPH), \x(2nd F), \x(STAT(Draw)), \x(5 T_Line), \x(ENTER)

Use the Calculator to read values of the graph for Y' on the curve $Y1 = X^2$ and complete the following table. Make sure that \x(TRACE) is on and that Y' is showing.
Use \x(2nd F), \x(ZOOM), \x(D_Y'), \x(1) \x(ENTER) to turn Y' on.

X	–4	–3	–2	–1	0	1	2	3	4
Y'									

What FUNCTION is represented by these values
\x(Y' =). Use the Calculator to check.
Graph Y1= 2X
Use \x(ZOOM) \x(7) to set increments at 0.1 and then switch to \x(TABLE) and check that the values agree with those in the table above.

Conclusion: Therefore we see that if $Y = X^2$ the Y' = 2X

Exercise: Plot the Function $Y1= X^3$

Use the Y' function to find values of the gradient (slope) of this function and fill in the table below.

X	−4	−3	−2	−1	0	1	2	3	4
Y'									

Guess a function rule.What function is represented by these values

\x(Y' =)

Test this by graphing the function \x(Y1) \x(2) \x(X/Q/T/N) and then checking the values in your table above against the \x(TABLE) on the calculator.

Write down a general rule for the gradient function of $y = x^n$.

Rule: \x(Y' =)

Check your rule for $y = x^4$ and $y = x^5$ by graphing the function, selecting values of y' and then using the \x(TABLE) on the calculator to test your prediction.

Extension: Work out gradient functions for the following and test your predictions using the calculator. 1. $y = x^2 + x$; 2. $y = 3x^2$; 3. $y = 2x^3 + x^2$

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Children's matching of melodies and line graphs

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Children (10-11 years) were asked to match short melodies and line graphs. Evidence was found for significant visual/graphical influences in the matching process (as well as previously established melodic factors). Intramodal tasks were performed better than cross-modal task, and within those categories, visual-first items were performed better than melody-first items. Mathematics ability was a significant main effect, but evidence pointed towards an effect of mathematics experience more than aptitude. Musical ability had only a limited effect.

Introduction

This research was designed to investigate children's matching of melodies with line graphs, with reference to the effects of modality, contour complexity, and musical and mathematical ability.

Morrongiello and Roes (1990) investigated children's ability to match short melodies with line graphs presented concurrently, and considered matching task performance in relation to contour complexity and melodic tonality, along with the age and musical training of the children. Their study showed that musically-trained children performed better at the matching tasks overall than musically-untrained children. However, for those musically trained, performance levels decreased with the use of atonal melodies in contrast to the consistent but lower performance levels of the musically-untrained children. These results indicate the sensitivity of the musical children to Western scale structure, and hence the importance of the melodic stimuli in the matching tasks. However, it is likely that the visual materials are just as important as the auditory materials in the matching process, and their effect should be investigated also.

To determine the extent to which visual stimuli have an influence on the matching process, this experiment tested children's sensitivity to conventional line graphs by comparing their performance using such graphs with their performance using non-conventional graphs. The non-conventional format comprised graphs with the vertical and horizontal axes rotated and reversed, such that up/down changes of pitch were represented by left/right changes of position, and time was represented vertically, from top down rather than horizontally from left to right. After considering the results of Morrongiello and Roes (1990) concerning tonal versus atonal melodies, it seemed reasonable to expect that, similarly, performance with conventional graphs would exceed performance with non-conventional graphs (the visual equivalent of atonal melodies). Also it seemed reasonable to expect that if the nature of the visual materials had a significant influence on task performance then there would be an interaction between contour complexity and visual format, such that increasing contour complexity would lower performance levels more with non-conventional line graphs than with conventional line graphs.

Balch and Muscatelli (1986) also investigated the matching of melodies and their visual representations; in their study, the visual representations were contour markers. They employed four modality conditions (visual to visual, visual to auditory, auditory to visual, auditory to auditory) and five presentation rates (from 10 notes per second to one note every two seconds). When melodies were presented at a rate of two notes per second or slower, significant differences emerged among modality conditions. For both low and high musically trained groups, intramodal visual to visual recognition was best, followed by recognition in both cross-modal conditions (visual to auditory and auditory to visual), and lowest performance occurred with the auditory to auditory condition. These results led Balch and Muscatelli to propose the *contour abstraction hypothesis* to the effect that the abstraction of up-down contour is more efficient from visual patterns than from auditory patterns. It is assumed that up/down contour abstraction takes time. Thus contour recognition should be difficult when presentation rate is extremely fast, regardless of modality condition. However as the presentation rate becomes slower, contour recognition would improve at different rates for different modality conditions producing an interaction between presentation rate and modality condition. The four modality conditions employed by Balch and Muscatelli (1986) were used in this experiment to extend the matching tasks involved in the study of Morrongiello and Roes (1990). The presentation rate selected for this experiment was 2 notes/second, the rate at which Balch and Muscatelli noted performance-level differences between modalities (on slowing the rate from 10 notes/second).

The contour abstraction hypothesis contrasts with the findings of some of the earlier research on the performance of various cross-modal and intramodal tasks. Although the results of research in this area from the 1960s and 1970s were not uniform, the general thrust was that cross-modal tasks are more difficult than intramodal tasks (Freides, 1974), including tasks involving auditory and visual modalities.

With the results of Balch and Muscatelli (1986) and the conclusions from modality studies implying otherwise, it was debatable as to which influences would be apparent with the tasks of this experiment under its particular conditions. If the contour abstraction hypothesis were applicable, then the order of performance would be the same as that observed by Balch and Muscatelli (1986), namely, visual-first tasks better than auditory-first tasks. Otherwise, it might be that intramodal tasks (visual-to-visual and auditory-to-auditory) would be performed better than cross-modal tasks (visual-to-auditory and auditory-to-visual).

With reference to contour complexity, the study by Morrongiello and Roes (1990) found that matching task performance levels generally decreased with higher levels of contour complexity. However the performance of musically-trained nine-year olds did not decline as complexity increased. In this experiment, it was expected that increasing contour complexity would have a detrimental effect on matching task performance overall. However, if musical ability has a positive influence on the perceptual processes in the matching tasks, then the musically-able children should be less influenced by increases in complexity compared to the musically less able children.

Both key studies referred to here (Morrongiello & Roes, 1990 and Balch & Muscatelli, 1986) identified musical training as a significant factor in the matching-task performance. The former study found that musically-trained children performed better than musically-untrained children at the matching tasks overall, as well as at the higher complexity levels. However, for those musically trained, performance levels decreased with the use of atonal melodies in contrast to the consistent but lower performance levels of the musically untrained children. Balch and Muscatelli also found that formal music experience facilitated matching-task performance overall for young adults. However, there was no interaction of music experience with modality condition nor was there an interaction of music experience with presentation rate. This raises questions concerning the role of the modality experience factor in the matching tasks. Work by Vande Voort, Senf and Benton (1972) and Goodnow (1971) suggests that it is not modality *per se* that matters in determining performance levels, but experience in the specific modality. The importance of experience with a particular modality and with specific tasks was highlighted further by Hershkovitz (1993) who found that children's ability at visual estimation was dependent on experience and instruction. If the variable, musical experience, does not affect the preferred modality conditions for the matching tasks, then perhaps musical ability does. Musical ability, as measured in this study, is likely to represent students with music experience in much the same way as those with aptitude but little experience. Hence musical ability includes aptitude and experience, and offers the prospect of confirming and extending the findings of Morrongiello and Roes (1990) and Balch and Muscatelli (1986) that were more obviously on experience/training. Hence musical ability was used as an alternative to music experience as a between-subject variable.

Mathematics training/experience would not have been a usefully discriminating variable to include in the study, because it would not vary greatly for children in any particular year level at school. They all would have studied mathematics for the same number of years, and would have spent similar numbers of hours per week at school engaged in mathematical activity. Hence, an experience-centred measure of mathematics would not be likely to apply cogently to the children in the study, and thus the need arose for a measure which was sensitive to aptitude and experience. Accordingly, it was decided to include mathematics ability as an individual difference variable rather than mathematics experience. The use of musical ability and mathematical meant that the music area and the spatial/mathematics area were treated comparably in this study.

Hence, tests of music ability and mathematical ability were constructed for the purpose of forming two sub-groups of children (high ability and low ability) for each area. The music ability test contained items on pitch and melody recognition and discrimination, and basic music notation. The mathematics ability test contained items on interpreting graphs, pattern recognition and measurement of length.

Method

Participants

Fifty-one children in Year 5 (aged between 10 and 11) from two regular suburban primary schools participated in the study. The sample contained approximately equal numbers of boys and girls; 38% of the children had been learning to play a musical instrument at the school or privately for at least one year. The formal music experiences of the other children had been one class-music lesson per week or less than one year of instrumental lessons plus one class-music lesson per week.

Design

The experiment was a 2 x 4 x 2 within-subject design (2 formats, 4 modality conditions and 2 complexity levels), combined (in separate analyses) with the between-subject variables of music ability and mathematical ability. Two visual formats were employed: conventional and non-conventional. The matching tasks required subjects to match melodies and line graphs in four modality conditions: auditory-to-visual, visual-to-auditory, visual-to-visual, and auditory-to-auditory. Melodies and line graphs were produced at two levels of complexity: low complexity with one contour change, and high complexity with three contour changes. The order of presentation of the experimental conditions was counterbalanced to distribute order effects (Burns, 1994).

Materials

The auditory materials were short tonal melodies, with nine notes of equal duration, played at a rate of 2 notes per second. They were generated on a Yamaha PSR7 keyboard using the familiar piano sound, taped by direct line on a Sony cassette recorder and subsequently played back to the children on a portable cassette player.

The visual materials of the matching task test were line graphs, each consisting of nine points joined by a single line. The line graphs (each measuring 16cm x 10cm) were printed black on white paper. The conventional line graphs were designed to be read from left to right, with up-down pitch movement being represented by vertical movement on the graph. In contrast, the non-conventional graphs were designed to be read from top to bottom, with corresponding up-down contour of melody represented by lateral movement on the graph: low on the right and high on the left. The materials were assembled into order in a spiral bound booklet for easy display and viewing. Half of the pairs of stimuli were matching pairs and half were not matching pairs. The pairs which did not match differed in terms of overall contour shape or the locations within the melody at which contour reversals occurred (start, middle, end).

The mathematics ability test consisted of 25 multiple-choice items on three mathematical topics: interpreting graphs, pattern recognition, and measurement of length. All items of the mathematics test were completed in a class situation. The written questions were also read to the class so that any children with reading difficulties would not be disadvantaged. The music ability test was a composite written and aural test of 25 items, some of which were multiple choice response items, and the others, short answer. Approximately half of the items in the test were completed in a class situation and the remainder were presented on an individual basis. The written items were also read to the class, as with the mathematics test, for the same reason.

Procedure

All children underwent the tests in mathematics ability and music ability initially and then completed all matching tasks. For the *melody-to-visual* modality condition, a melody was played on the cassette player to the child, and then a line graph was shown for approximately four to five seconds (long enough for the child to indicate that he/she had noted its features, and for a duration comparable to the length of time required to hear the nine-note melody). The question was asked, "Is this graph the same as the melody you have just heard, or is it different?" In the *visual-to-melody* modality condition, a line graph was shown to the child for four to five seconds. Then a nine-note melody was played. The child was asked, "Was that melody the same as the graph on the page before, or different?" In the *visual-to-visual* modality condition, a line graph was shown to the child, then the page was turned to reveal another line graph. The child was asked if the two graphs were the same or different. In the *melody-to-melody* condition, two melodies were played consecutively and the child was asked if the two melodies were the same or different.

Results and discussion

The proportion of correct matching responses was determined on the basis of the number of correctly identified matches (hits) plus the number of correctly identified mismatches (correct rejections), the latter being included to reduce the undesirable effects of false alarms, and also to conform to the basis used in the study by Balch and Muscatelli (1986); i.e. p(correct) = (no. hits + no. correct rejections) / (no. matches + no. mismatches).

Mean proportions correct were calculated for levels of matching task performance in each experimental condition for the whole sample and for each ability group: high/low mathematics ability and high/low music ability. Ability groups were determined on the basis of median splits on the aggregate test scores.

There were four major issues investigated in this experiment -
(i) the extent of visual/graphical influences in the melody-visual matching process; (ii) the claim that contour abstraction is more successful from visual materials than from auditory materials; (iii) the effect of ability factors on the order of task-performance levels of the various modality conditions; and (iv) the effect of ability factors on the perceptual processes involved in the matching process. Results relating to these major issues are now presented in turn.

The extent of visual/graphical influences in the matching process

Evidence was found for significant visual/graphical influences in the matching process. Manipulation of the visual materials in terms of conventional and non-conventional format produced different levels of performance, significantly with visual-to-melody and melody-to-visual tasks. There were significant main effects for format, $F(1, 49) = 8.74$, $p < .01$, $MSE = 0.067$, (conventional better than non-conventional), complexity, $F(1, 49) = 22.61$, $p = .0001$, $MSE = 0.039$, (low better than high), and modality, $F(2, 98) = 92.33$, $p = .0001$, $MSE = 0.072$, (visual-to-visual better than visual-to-melody, in turn, better than melody-to-visual; melody-to-melody was approximately equal to visual-to-melody in the four-modality condition model, Table 2). However, the significant format x complexity interaction, $F(1, 49) = 20.56$, $p = .0001$, $MSE = 0.04$, qualified the first two main effects such that increasing the complexity level reduced performance of tasks involving conventional line graphs more than performance of tasks involving non-conventional line graphs, performance on which was relatively weak. Clearly, the matching process is influenced not only by melodic/auditory factors (as assumed implicitly by Morrongiello & Roes, 1990) but also by visual/graphical factors.

The claim that contour abstraction is more successful from visual materials than from auditory materials

The superiority of visual-first items as proposed by Balch and Muscatelli (1986) was not supported. Instead, it was found that performance was determined by two considerations - (i) intramodal tasks were performed better than cross-modal tasks, consistent with much of the modality literature (Freides, 1974), and (ii) within each of the modality-condition categories, intramodal and cross-modal, visual-first items were superior to auditory-first items.

The observed effects of modality and format lead to the development of a set of theoretical principles for the auditory/visual matching process, which proposed that matching entailed a comparison of two specific components, (i) an abstraction of the first-presented stimulus and (ii) the second stimulus. Abstractions of visual contour differ from abstractions of auditory contour in that visual coding differs from auditory coding. The contour information abstracted preserves modality information as well as the up/down sequence information. In the case of intramodal matching, as the two stimuli are of the same modality, the second stimulus is compared with an abstraction of the first stimulus. In the case of cross-modal tasks, before a comparison can be made, a conversion (or recoding) step is required to allow for the matching of 'like' modalities. For example, in a visual-to-melody task, the contour first is abstracted from the line graph. Then for a comparison to be made with the melody, either the contour is abstracted from the melody and then converted (recoded) to its visual equivalent for matching, or the visually abstracted contour is converted (recoded) to an auditory equivalent for comparison with the melody as it is played. It seems reasonable that this step in the process would vary from child to child, depending on his/her ability and experience. This extra complication in the process reduces the likelihood of a successful match being made - hence the lower performance levels of cross-modal tasks.

The effect of ability factors on the order of task-performance levels of the various modality conditions

Ability factors had little effect on performance under the modality conditions. Although musical ability, music experience and mathematical ability had significant effects on performance levels of tasks in certain modality conditions, these effects were not as great as the effects of modality, format, and complexity.

The effect of ability factors on the perceptual processes involved in the matching tasks

Mathematics ability had a significant positive effect on performance levels overall, $F(1, 49) = 4.34$, $p < .05$, $MSE = .097$, but investigation of the results of the two visual formats separately revealed that the effect of mathematics ability was significant for the tasks involving conventional line graphs, $F(1, 49) = 5.31$, $p < .05$, $MSE = .077$, but not for tasks involving non-conventional line graphs, $F(1, 49) = 1.53$, $p > .05$, $MSE = .093$. In other words, the advantage that the high mathematics ability group had over their low ability counterparts with tasks involving conventional line graphs disappeared when non-conventional line graphs were involved. This result suggests that the superior performance of the high mathematics ability group can be attributed to some extent to that group's greater familiarity and with the conventional line-graph format, and hence is evidence of an experience effect, rather than an effect of natural aptitude. Given the overlap between mathematics ability and mathematics experience, it is likely that a test of mathematical ability is also a measure, to some degree, of mathematics experience. Hence the positive effect of mathematics ability on performance levels found in this experiment is evidence of the contribution experience makes to the ability measure.

In contrast to the effect of mathematics ability, music ability had no significant effect on performance overall, $F(1, 49) = 2.24$, $p > .05$, $MSE = .124$, on performance with tasks involving conventional format, $F(1, 49) = 3.68$, $p > .05$, $MSE = .079$, nor on performance with tasks involving non-conventional format, $F(1, 49) = 4.44$, $p > .05$, $MSE = .095$. However, music ability did have a significant main effect in one particular condition, namely, high-complexity, visual-to-melody items involving conventional format in which high-ability children performed at a higher level than their low-ability counterparts, $F(1, 49) = 7.15$, $p = .01$, $MSE = .130$. This instance of a positive music-ability effect is the only one that corresponds with the results of Morrongiello and Roes (1990) who demonstrated that musical training was a significant positive factor in matching performance levels. The discrepancies with the other conditions (e.g. low and high complexity melody-to-visual tasks) may be explained by differences in mode of presentation and the nature of the tasks. This experiment used consecutive presentation of auditory and visual materials, whereas Morrongiello and Roes used concurrent presentation. The consecutive mode of presentation requires an element of short-term memory which may have caused greater difficulty for many of the children in some conditions of the present study, both musical and non-musical. Morrongiello and Roes noted that many of their musically-trained children actually watched the line graph intently as the melody was played, and some nodded their heads up and down in phase with the contour movement. This strategy is possible for concurrent presentation, but not for consecutive presentation, and indicates that these particular children were making a point-to-point comparison of melody and line graph. It is also consistent with an effect of music experience rather than natural aptitude, and therefore suggests that music experience is an important component of the ability measure.

Moreover, the nature of the tasks differed with respect to the visual materials used and the response required of the children. In this experiment, a new pair of stimuli was presented to the children for each item and they were asked to indicate whether the stimuli matched or not, whereas Morrongiello and Roes (1990) had just three standard visual stimuli from which the children could select one which they thought matched the melody heard. On one hand, the presence of the three graphs in the Morrongiello and Roes procedure may have simplified the tasks because the children could be sure that one of the graphs did match the melody. However, on the other hand, it may have complicated the tasks if the first graph observed did not seem to match the melody, because a child would have had to shift visual focus to a second graph while the melody was still playing, and possibly to a third. The children least likely to be distracted by this complication would be the musically-trained children, those who were experienced in associating a melody with a visual representation of it.

Conclusion

This study demonstrated that children's ability to match melodies with line graphs is determined not only by auditory factors (as found by Morrongiello & Roes, 1990) but also by visual factors. Just as matching-task performance is affected by a change in musical scale structure (from tonal to atonal), so too is it affected by a change of visual format (from conventional to non-conventional). The contrast of the results of this study (in which the visual materials were line grapsh) with those of Balch and Muscatelli (1986) (in which the visual materials were contour markers) also suggests that performance levels are influenced by the nature of the visual materials. This effect could be further investigated by the use another visual representation, one which is particularly relevant to the study of melodies, namely conventional music notation. Future research could investigate the matching process with the use of both conventional music notation and a non-conventional notation.

The study also showed that the effect of modality on matching-task performance is not as clear-cut as claimed by Balch and Muscatelli (1986). The claim of the superiority of visual-first items did not hold true for the materials used with Year 5 school children. In this study, visual-first was a second consideration after intramodal versus cross-modal considerations. Would it be so for matching tasks involving music notation too?

The effect of musical ability on the perceptual processes involved in the matching tasks was less marked than that exhibited in the studies by Morrongiello and Roes (1990) and Balch and Muscatelli (1986). However, in the light of the demonstrated effect of mathematics ability in the present study, it seems feasible that in the future research alluded to in the above paragraph (matching melodies with music notation) an effect of music ability might be shown rather than mathematical ability.

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Identifying Separate and Connected Knowing in Mathematics Education

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This paper describes the development of a questionnaire to identify Separate and Connected Knowers (Belenky, Goldberger, Clinchy and Tarule (1986) in mathematics. Justification for the selection of items is provided from the work of Belenky et al (1986), Buerk (1985), Becker (1995, 1996), Koch (1996), and Erchick (1996). The questionnaire was administered to 67 university students as part of a larger study. Interviews within the main study showed that the questionnaire had served its purpose; it had accurately identified a group of Separate Knowers in mathematics.

Introduction

The purpose of the main study referred to within in this paper was to investigate the influence of moral orientation (Gilligan, 1982) on participation in traditional mathematics education. However, in this paper I report only on the methodology used to generate the sample for the main study interviews. The influence of moral orientation on participation in mathematics education has been discussed in a preliminary way elsewhere (Ocean, 1997).

Over the last decade, Belenky, Goldberger, Clincy and Tarule's description of Women's Ways of Knowing has been adapted to mathematics education by Buerk (1985); Becker (1995, 1996); Morrow and Morrow (1995); Koch, (1996); Erchick (1996), Morrow (1996), and Boaler (1997a, 1997b). These authors describe the Separate Knowing approach to mathematics as rule-based, memory-reliant, individualistic and competitive. In contrast, a Connected Knowing approach to mathematics is concept-based and creative, and places the emphasis on co-operation rather than competition. Such an approach has been found to produce greater success for girls (Boaler, 1997a) and women (Rogers, 1990). Erchick expresses one of the concerns of those who write in the area of Connected Knowing in mathematics: "If mathematics is (or is perceived to be) a formal system that threatens connectedness, a language that expresses power-over and control, a masculine space, a space that disallows subjectivism, perhaps most women really would choose not to participate in it" (Erchick, 1996, p.120). That is, if mathematics is presented only as Separate Knowing (sometimes called "traditional" mathematics education in this paper), the attraction and retention of female students will be low.

The questionnaire was designed to identify individuals who had had an extreme Separate Knowing experience of mathematics education at school. It identified nine such individuals from a sample of 67 third- and fourth-year university students enrolled in engineering and education courses. Subsequent interviews with six of these students confirmed the questionnaire's usefulness as an identifier of a Separate Knowing school mathematics education. No claims are made here for any wider application of the questionnaire than the generation of a sample for later interviews. However it is suggested that the questionnaire, because it did serve its purpose, may be useful as a basis for further work in this area.

Women's Ways of Knowing

Women's Ways of Knowing was developed into a theory of intellectual development by Belenky, Goldberger, Clinchy and Tarule (1986). Prompted by Perry's (1970) widely-accepted model of intellectual development, which used only interviews with men to validate the model, they asked what else women might have to say about their own intellectual development. They conducted interviews over an extended period of time with 135 women of diverse ages, races, and circumstances.As a result, Belenky et al posited five "stages of knowing"; silent knowing, received knowing, subjective knowing, procedural knowing, and constructed knowing. These stages are not necessarily developmental; that is, women do not necessarily pass through them in this order, or through all of them.

The five stages of knowing were expanded with reference to mathematics education by Erchick (1996). Erchick's work will now be referred to in detail to describe each stage. This will be followed by Table 1, which draws on the work of Belenky et al (1986), Becker (1995, 1996), Koch (1996), Buerk (1985), and Erchick (1996) to further illustrate each of the stages.

Women's Ways of Knowing in Mathematics

In interviewing women who were primary school teachers about their experiences of mathematics, Erchick repeatedly heard talk of the *silent* way of *knowing*. These women asked no questions, and few asked teachers for help. They felt voiceless, and stayed silent regarding mathematics until well into adulthood. Erchick suggests that the silent learner may fit rather well into the mathematics classroom, especially in the primary school years, and "may even be rewarded for their silence with praise for being a good student" (Erchick, 1996, p. 112).

In contrast, the *received knower* depends on words; she learns by listening. But she listens to the voice of an authority - to her, there is only one right answer that the teacher will dispense. The formal system of mathematics, with its "one way to do things" should feel comfortably familiar to women in this way of knowing (Erchick; 1996, p. 112). This type of knower was evident in Erchick's (1994) interviews with primary school teachers, with one woman, for example, describing her expectation that simply attending all classes, listening, and taking notes would be sufficient to produce learning (Erchick, 1996, p. 113).

With the *subjective knower*, it is not only the authority who knows. With this perspective, women begin to become their own authorities. Belenky et al (1986) found that women move into this perspective as a result of having encountered a failed male authority. They still depend on authorities, from and with whom they learn a truth that they know and share, but these are usually now female authorities, more like themselves. For the subjective knower, truth is a private and individual matter for everyone. Abstraction, logic and analysis, which might produce common truths, are distrusted (Erchick, 1996, p. 114).

Procedural Knowing consists of two kinds of knowing, Separate Knowing and Connected Knowing. These are not regarded as gender-specific, but they may be gender-related - the preferred styles of men and women respectively (Belenky et al, 1986). For the connected knower, understanding is predominant- for the separate knower, knowledge is predominant. The model of Separate Knowing closely reflects the individualist and competitive values of traditional mathematics education, while Connected Knowing is more representative of a style that places emphasis on co-operation, discussion, and group work. Erchick (1996) observes that the connected knower finds both expertise and constructive criticism within the group, and so finds it helpful to maintain a group connection. The separate knower, on the other hand, is involved with others more as competitors than collaborators. She learns to distance the self from the object of knowledge. Certainty, mastery and control are features of separate knowing.

The key to *constructed knowledge* lies in the integration of thought, feeling, and experience (Erchick, 1996). The constructed knower can call on both connected and separate knowing when appropriate to create or recall the mathematics that she needs.

Table 1 provides a summary of these Ways of Knowing in mathematics, with a description of how the knower might behave and an example of what the knower might say in the mathematics classroom.

Table 1 Women's Ways of KnowingIn Mathematics Education

(drawn from Belenky et al (1986), Buerk (1985), Becker (1995, 1996), Koch (1996), Erchick (1996)).

Ways of Knowing	The Knower	The Knower's Voice
Silent Knowing In the first stage knowing is subliminal. It is not articulated, and the knower does not believe that she can learn from her own experience (Becker, 1995).	She does not speak, and does not expect reasons to be given. She does not believe that she would understand any explanation given (Erchick, 1996).	Silent knowers rarely speak. They may be rewarded for their silence with praise for being a good student (Erchick, 1996).
Received Knowing Received Knowers believe they are only able to receive and reproduce knowledge and not to produce knowledge on their own. Knowledge emanates from external authorities (Koch, 1996).	The received knower will feel confused if asked to do original work (Erchick, 1996). A received knower is not ready to give up her belief that mathematical knowledge comes from the teacher and that mathematics is a series of calculations (Koch, 1996).	"How could she learn if he wouldn't pass along the answers?" (Belenky et al, 1986, p.40)
Subjective Knowing Authority locates within the knower - knowledge derives from what "feels right". For a subjective knower, there is no place for the integration of contradictory evidence - it is ignored. There is absolute truth, but it exists only for the individual (whereas the silent and received knowers believe it to be universal) (Erchick, 1996).	Students might rely on other students who show them how to do the work. These students may be seen as authorities even though they might have failed the course previously. The experience of difficulty and effort may be the common bond between a subjectivist learner and the new authorities who would be people very like the student herself (Erchick, 1996).	"I know that base angles on an isosceles triangle are equal. Just look at them: they're equal" (Becker, 1995, p. 163).

Procedural Knowing consists of two kinds of knowing, Separate Knowing and Connected Knowing. These are not regarded as gender-specific, but they may be gender-related - the preferred styles of men and women respectively (Belenky et al, 1986). **Separate Knowing** Separate Knowing is based on the use of impersonal procedures to establish truths (Becker, 1995). The Separate Knowing approach gets right to the solution in a structured, algorithmic way, stripping away any context (Buerk, 1985). **Connected Knowing** Connected Knowers build on personal experiences. Context is important. A Connected Knower will want to know what circumstances led you to your perception, rather than what logical steps led to your conclusion (Becker, 1985). Connected knowing is complex, related, considers many things simultaneously.	**Separate Knowing** In problem solving, for example, a problem would be solved repeatedly by many examples solved exactly the same way. **Connected Knowing** In problem solving, for example, the same problem would be solved in different ways, rather than additional examples solved exactly the same way.	**Separate Knowing** Example : "the main thing that's nice about math is: that's it. You do what it says right there. I like that...one can read a Shakespeare play for the rest of one's life and not have any definite idea of what's going on" (Becker, 1996, p. 22). **Connected Knowing** Example : "(Mathematics) is beautiful. It ties in with so many things, it encompasses so many things" (Becker, 1996, p. 23)
Constructed Knowing When an individual has learned both separate and connected ways of knowing, and can use each appropriately, this is called Constructed Knowing. In this stage answers are dependent on the context in which questions are asked and on the frame of reference of the asker (Becker, 1995).	Constructed Knowers are able to use both Separate Knowing and Connected Knowing when required, in solving problems. "The solution of an equation is dependent on the domain being considered, and what can be proven is dependent on the axioms being assumed...here, the knower would use the creative aspect, induction, together with the rules of discourse, deduction, in order to know something" (Becker, 1995, p.168).	"Let's physically compare the angles" (Becker, 1995, p. 165). "Tell me why you think that base angles are equal" (Becker, 1995, p. 165).

Developing The Questionnaire

In the pilot for the main study (not reported here - see Ocean, 1997), I had interviewed six adult women (friends and colleagues) to investigate the influence of moral orientation on participation in traditional mathematics education. I had obtained a sample by making requests of friends and colleagues who really disliked mathematics education and who I thought might provide rich data. This approach to generating a sample produced a particular problem, however, that an example will illustrate. One of the participants, Sue, turned out to be a silent/received knower. Sue viewed mathematics as entirely beyond her, and regarded that as entirely her fault. She was difficult to interview because she didn't see herself as participating in mathematics in any way - in effect, I got very little data from her at all. In the main study, I wanted to avoid this pitfall, and so developed the questionnaire to identify separate knowers and connected knowers. I thought they would (a) actually know more mathematics and (b) be more articulate about their own mathematics education than silent, received, or subjective knowers. I subsequently decided to interview only the Separate Knowers.

Methodology

Fifteen items chosen to identify Separate Knowing in mathematics education were taken from Schoenfeld's (1989) unrelated questionnaire *Explorations of Students' Mathematical Beliefs and Behavior*. The advantage of selecting items from an existing questionnaire are that some of the possible pitfalls of questionnaire design, such as ambiguous language, wording effects, and question order (Loewenthal, 1996) have already been addressed. Justification for the selection of items from Schoenfeld's questionnaire will be given shortly.

Schoenfeld's questionnaire contains 81 items. It surveys attributions of success and failure, perceptions of mathematics and of school practice, the nature of (geometric) proofs, reasoning and constructions, motivation, and personal and scholastic performance (Schoenfeld, 1989, p. 342). While the items were designed for senior high-school students, the language was suitable for adults. Minor changes were made to better fit some items to general mathematics, and to Australian English. For instance, the most extensive change was made to the item *(When I do a geometry proof), I'm finished if I can't remember the next step*. This was altered to *(When I solve maths problems), I'm often stuck if I can't remember the next step*.

Fifteen multiple-choice items amongst the eighty-one in the questionnaire corresponded principally to either Separate Knowing or to Connected Knowing. These fifteen items were sent to a reviewer who had completed a PhD on Separate and Connected Knowing two years previously. She was asked to rate each item. Inter-rater agreement for the fifteen items was 100%.

The questionnaire asked participants for their view of school mathematics. They were asked to disregard their experience of university mathematics, if they had any, when answering. Responses were asked for on a Likert-type scale, with participants being asked to strongly agree, agree, remain undecided, disagree, or strongly disagree with each item. While no more weight was given in the analysis to the stronger comment, five rather than three options provides more choice to the respondent. Space was left under each item for additional comment.

Items that were chosen for the questionnaire as representative of Separate or Connected Knowing reflect the descriptors found in Table 2 (Ways of Knowing in Mathematics). Recalling Erchick's (1996) comment that mathematics may be perceived as a formal, controlling, masculine space, with no room for subjectivism, Separate Knowing in mathematics involves facts, formulae, known procedures, reliance on memory, certainty (right vs wrong), lack of creativity, and no acceptance of alternative methods for solution.Mathematics is seen as an absolute and finite body of knowledge. Within Connected Knowing, mathematics involves creativity, conceptual understanding, complexity, context, inter-relatedness of ideas, and acceptance of alternative methods for solution.

It should be noted that items chosen to represent Separate Knowing may have been selected on the questionnaire by a silent, received or subjective knower. Alternatively, these knowers may have selected the 'undecided' option provided. Correspondingly, items that were chosen to reflect Connected Knowing may have been selected by a constructed knower. Results from the questionnaire should be read with this in mind. While it did not affect the usefulness of the questionnaire in my study (because a silent or received or subjective knower is still likely to have had a "traditional" mathematics education, and similarly a constructed knower will have been exposed to a connected knowing approach to mathematics), it may be an important point in any further application of the questionnaire.

Table 2 links each of the fifteen multiple-choice questionnaire items with the relevant descriptors of Separate Knowing and Connected Knowing.

Table 2: The Links Between the Questionnaire items and Indicators of Separate Knowing and Connected Knowing in Mathematics.

Q1. Maths is mostly facts and procedures that have to be remembered. Indicators of Separate Knowing: Reliance on memory, facts, formulas and procedures.

Q2. Maths makes you think creatively. Indicator of Connected Knowing: Creativity.

Q3. When the teacher asks a question in maths, you have to remember the right answer to answer it correctly. Indicators of Separate Knowing: Reliance on memory, certainty.

Q4. When the teacher asks a question in maths, there are lots of possible answers you might give. Indicators of Connected Knowing: Opportunity for creative input, context, alternative paths to solution.

Q5. Good maths teachers show you the exact way to answer the mathematics questions you have to do. Indicators of Separate Knowing: Limited or no opportunity for creative input.

Q6. In maths, something is either right or it's wrong. Indicator of Separate Knowing: Certainty.

Q7.Good maths teachers show students several different ways to look at the same question. Indicator of Connected Knowing: Alternative paths to solution are possible, creativity.

Q8. Everything important about maths is already known. Indicators of Separate Knowing:Limited or no Opportunity for creative input, maths is absolute and finite.

Q9. In maths you can be creative and discover things by yourself. Indicator of Connected Knowing: Creativity.

Q10. Some people are good at maths and some just aren't. Indicator of Separate Knowing:Certainty.

Q11. Problems can be done correctly in only one way. Indicator of Separate Knowing:Certainty, maths is absolute and finite.

Q12. To solve maths problems you have to be taught the right procedure or you can't do anything. Indicator of Separate Knowing: Certainty, maths is absolute and finite.

Q13. The best way to do well in maths is to remember all the formulas. Indicators of Separate Knowing: Reliance on memory , facts, formulas and procedures.

Q14.When I solve maths problems, I can only prove something a mathematician has already shown to be true. Indicators of Separate Knowing: Limited or no opportunity for creative input, Certainty.

Q15. When I solve maths problems, I'm often stuck if I can't remember the next step. Indicators of Separate Knowing: Reliance on memory, facts, formulas and procedures.

Questionnaire Administration: The questionnaire was administered to 46 (fourth-year) graduate Diploma in Education students, and 21 (third-year) Electronic Engineering students, in class time. It took about twenty minutes to complete. All volunteers read an information sheet and gave their consent in writing. Refusal rates were 4/44 Diploma in Education (Primary), 8/14 Diploma in Education (Secondary), and 13/33 Engineering.

Questionnaire Analysis: Each response was scored as either undecided or Separate Knowing (S) or Connected Knowing (C). Agreement with a statement originally chosen to represent Connected Knowing was given a C score, while disagreement with the statement was given an S score, and vice versa. Undecided responses were not counted in the total. As the number of undecided responses varied between students, a student's view of mathematics was labelled S or C if 75% or more of their responses fell into that category. Otherwise, their responses were assigned to the combined (S/C) category. This produced small groups of people at either extreme, which suited the questionnaire's purpose of identifying an extreme sample. Subsequently it was decided only to interview those with an extreme Separate Knowing mathematics education.Table 3 shows the distributions of the scores in total, and across courses.

Results and Discussion

Table 3: Distribution of Ways of Knowing by Course of Study

Course of Study	Separate Knowing	Connected Knowing	S+C Knowing
Engineering (n=21)	3	3	15
Dip. Ed (Primary)(n=40)	6	11	23
Dip. Ed (Sec) (n=6)	0	1	5
Total (n=67)	9 (13%)	15 (22%)	43 (64%)

The questionnaire thus generated a pool of nine individuals who had a view of mathematics that was at the extreme end of the individualist, rule-based, competitive - "traditional" - approach to mathematics education. Six of these were later interviewed in depth in semi-structured interviews, primarily to seek their moral response to their experience of traditional mathematics education. However, the Separate Knowing categorisation of each student from the questionnaire was cross-checked in these interviews. This was done in two ways; by (a) asking questions in the interview about what mathematics was like for them at school, and listening for a Separate or Connected Knowing description of mathematics at school, and (b) by using a number of written descriptors of Separate and Connected Knowing as a prompt for a discussion of their view of mathematics. These descriptors were taken from the work of Buerk (1985); Becker (1995, 1996); Morrow and Morrow (1995); Koch, (1996); Erchick (1996), and Morrow (1996) . They were written on cards, and participants chose those that *most* and *least* represented mathematics to them. For Separate Knowing the descriptors included absolute, finite, rules, algorithms, just and fair. For Connected Knowing, the descriptors included connected, intuitive, contextual, tolerant, models and creative.

This cross-check was an additional attempt to ensure that I was indeed listening to students who had a Separate Knowing mathematics education. It was not intended as a check on the validity of the questionnaire, as the numbers checked (6 out of 67) are too small. However, all six students did have a Separate Knowing approach to mathematics. None had a silent, received, subjective or constructed way of knowing. Because the questionnaire proved an accurate predictor of the participant's high school mathematics experience for the six individuals in this study, it very adequately served the purpose for which it had been designed. No claims are made here for its wider application. However, since it affirmed rather than contradicted data obtained from interviews some researchers may think it worthwhile as a starting point for further development of items to assess Separate and Connected Knowing in mathematics. For instance, following a re-examination of the "Mathematics as a Male Domain" scale (Forgasz, Leder, and Gardner (in press), these researchers are trialing some of the questions related to Separate and Connected Knowing drawn from this questionnaire (personal communication, Leder and Forgasz, 1998).

Conclusion

This paper describes the development and implementation of a questionnaire which was designed to identify Separate and Connected Knowing in mathematics, drawing on the work of Belenky, Goldberger, Clinchy and Tarule (1986), Becker (1995, 1996), Koch (1996), Buerk (1985), Erchick (1996) and Schoenfeld (1989). It proved effective in identifying a sample of students who had had a traditional Separate Knowing mathematics education. No claims are made for its wider application than the generation of a sample for in-depth interviews. However, it may be of use as a starting point in the development of further questionnaires to measure Separate and Connected Knowing in mathematics education.

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Developing the angle concept through investigations

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Children in Year 2 and 4 were developing the concept of angle through their responsiveness, manipulating of materials, and interactions with others. Their selective attention to aspects of the materials, interactions, or their own imagery and thinking assisted the development of the concept.

Introduction

Piaget and Inhelder (1956) concluded from their studies that there is no doubt

> that it is the analysis of the angle which marks the transition from topological relationships to the perception of Euclidean ones. It is not the straight line itself which the child contrasts with round shapes, but rather that conjunction of straight lines which go to form an angle. (p. 30)

Students intuitively recognise an angle at an early age but need to develop the concept, associated language, imagery, angle contexts, and related constructs such as size of angle.

Children (e.g. Years 3 and 6) have difficulty in recognising equal angles in differing orientations, with differing arm lengths, and when embedded in figures (Outhred, 1987). Outhred found that right angles that were not aligned with the horizontal axis were often not recognised. Similar difficulties have been mentioned in reports of other studies (Fuys, Geddes, & Tischler, 1988; Mitchelmore, 1989; Pegg & Davey, 1989). More pre-measurement informal experiences with angles, their size, orientation, and contexts were recommended by Close (1982). The pre-measurement activities used in the current study encourage perception of angles in figures.

The ideas of *being pointy* and of *being a corner* or right angle is understood relatively early (Davey & Pegg, 1991). The right angle is not regarded as an angle by some students, others believe that only right angles are angles, and others think that only acute angles are angles. Teaching activities need to help children see right angles as special angles (Mitchelmore, 1992). The present study placed right angles in the family of angles and encouraged informal comparison of angles.

Year 2 and 4 children have a good global understanding of different angle contexts, and they improve significantly in their recognition of angle-related similarities (Mitchelmore,1993, 1994). Children most easily recognise similarities between situations where both lines are physically present (e.g., crossing) and are nearly as able when one line is present and one has to be imagined (e.g. a slope), but they find it difficult to relate turning, for example, of the body to face from one direction to another, (both lines to be imagined) to other angle contexts. Mitchelmore and White (1995) explained the development of general angle concepts in terms of abstraction. The angle concept is thought to develop gradually as children recognise more and deeper similarities between physical angle experiences, going through three successive stages: classification into physical angle situations such as walking up a slope and using scissors; then into separate angle contexts described as *sloping* and *crossing*; and finally into a general angle concept which includes all contexts. Turning, according to Mitchelmore, was more likely to be interpreted as a static anglerather than direction turning. In the present study, turning was related to static angles of shapes and other angle contexts were not used.

Abstractions are generally associated with concept imagery which needs to be dynamic and multifaceted if it is to encapsulate the concept adequately (Battista & Clements, 1991; Owens, 1994). The importance of mental imagery in estimation of angle size was illustrated by sixth and eighth grade students who imagined a protractor, a right angle, a half turn, or an angle of a polygon (Mansfield & Happs, 1992).

For two questions on angles, one on recognising a straight angle and the other on the angle of slope to the horizontal, only 74% and 64% respectively of Year 6 students New South Wales were successful (Owens, 1997). These data suggest that it is apparent that students have difficulties recognising angles in everyday situations, although it is recognised that the results could indicate that students have difficulties understanding the words: *straight angle* (*straight up or straight angle* is sometimes used for horizontally oriented right angles) and *horizontal*. Language difficulties were apparent when sixth grade students gave descriptions using non-standard vocabulary during Fuys et al.'s (1988) study although more formal language had been used in class.

Language difficulties make it hard to study the development of the concept of angle. Some researchers (e.g., Fuys et al., 1988; Pegg & Davey, 1989; van Hiele, 1986) have claimed that the type of language and the concept images which students use relate to a particular level(e.g. Van Hiele, 1986, levels). Language, however, can act as a mediator to differentiate a particular stimulus and promote attention. The connections between language and image may not be tied by level but, indeed, images and language could be regarded as limited conceptions promoting attention. The study to be described in this paper illustrated the importance of language in concept development and in focusing attention during problem solving and learning. There was a language difficulty also in using the word *angle* with young children because it seemed to have no meaning from earlier experiences. In an attempt to overcome this problem the word *point* was also used to refer to angle. However, this sometimes limited children's thinking about angles.

Clements and Battista (1992) have argued that students' learning can be explained in terms of cognitive processing rather than in terms of levels of development. It is, therefore, important that a qualitative study develop an understanding of students' conceptual learning by allowing the theoretical perspective to emerge from the data of the study.

The Study

The study focused on how students learnt from spatial problem-solving activities. The activities were designed to promote spatial thinking, especially imagery, problem solving, and analysis. Several concepts were developed during the study and this report will focus on the development of the concept of angles.

The qualitative study involved students in Years 2 and 4 in Australia and Papua New Guinea. The activities given to the students during 11 sessions were open-ended spatial problem-solving activities which improved students two-dimensional spatial thinking (Owens, 1993a). The type of problem-solving experiences encountered by the students can be illustrated by those posed for the well-known seven-piece tangram set made of three-sizes of isosceles right-angled triangles, a square, and a parallelogram (see Figure 1). Students were to look for similarities and differences between the pieces, to make the larger pieces from the smaller pieces, to make different-sized squares, triangles, and rectangles. Later they were asked to order the angles in size (small, middle-sized, and large) and make the shapes with sticks and matchsticks. Similar activities were completed using pattern-block sets containing squares, equilateral triangles, isosceles trapezia and two types of rhombi.

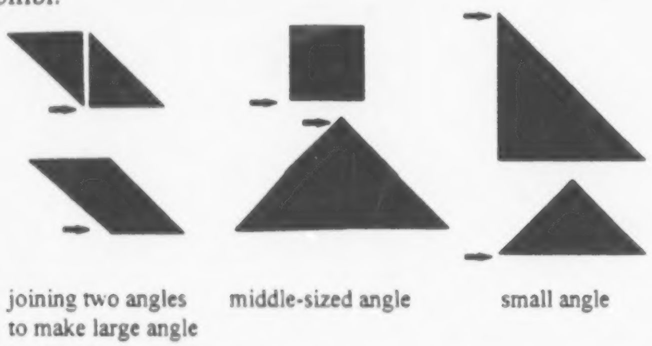

Figure 1. The angles of the seven-piece tangram set.

Students were assisted in recognising angles by turning their first finger, away from their thumb to mark the angles on the pieces as illustrated in Figure 2.

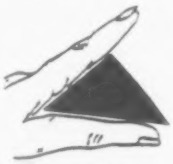

Figure 2. Using thumb and finger to note an angle on a tangram piece.

Several groups of students were purposefully selected as the qualitative aspects of learning unfolded. Several categories of thinking such as imagining and affect were defined as important from the initial sample of students who worked by themselves and provided immediate retrospective recall and spontaneous comments (Owens, 1991).

In the next stage, the effects of interaction were considered. Two groups of three children in Year 2 and two in Year 4 were given the problems and stimulated to recall how they were thinking immediately after solving each problem. One group from each year worked as a cooperative small group and the other as individuals but able to talk to their peers. Classroom contexts were considered when similar groups were used in classes in three schools from one region of Sydney, in a school from another suburban region, and in another cultural setting, namely in Papua New Guinea.

Students were video-taped while engaged in the activities and these tapes were analysed. Each incident during which there was a small development in problem solving was categorised. The categories included interaction with materials, student-student and student-teacher interaction, concepts, imagery, heuristics, and affect. The subcategories were developed during the grounded-theory research and linked to the literature review.

Results and Discussion

Examples have been chosen to illustrate how a concept, in this case that of an angle, can be developed in terms that are not associated with levels of development but with cognitive processing, and influences of the context of learning.

For Dora, in Year 2, there was a conflict between her perception of the materials and what she had observed a fellow student doing. Dora set out to compare the angles of the tangram pieces. The following notes described her participation in the activity.

1.01 Dora tells her teacher that her fingers have to be spread further apart for the large angle of the parallelogram. She gathers together most of the right angles. She puts her thumb and finger around them.

1.02 She seems a little exasperated for she is unsure about what is expected of them. She discusses with her friend about recording the angles—small, middle, and large—in her book and she draws the two arms of the small angle.

1.03 She knows that, beside the small triangle, both the big triangle and the parallelogram have a small angle.

1.04 She compares the right angle with the small angle but her friend calls the right angle large, so the teacher reassures her that the right angle is in the middle....

1.05 When the teacher asks her for the large angle, she picks up the parallelogram and claims she had it before her friend....

1.06 She joins up angles to make a right angle, and is pleased about doing this.

1.07 Then she joins the two triangles to make a parallelogram ... and shows this to the teacher as the large angle. (See Figure 1 configurations)

It is clear that this Year 2 child had perceived the differences between different angles on the various shapes. She was able to relate size of the angle to the position of the arms of the angle. The concrete materials helped her to see the same angle in different shapes (paragraphs 1.01, 1.03, and 1.04), and she was able to mark the angles with her fingers and to draw them (paragraph 1.02). She also compared angles with other angles represented by the pieces and drawings (paragraph 1.04). The above account also draws attention to a common affective response to making shapes or angles, that of pleasure and excitement following a measure of success with an activity (paragraphs 1.06 and 1.07). It is as if the concrete materials were often being used by students as a means of confirming and celebrating their abstract idea of an angle.

The materials provided physical representations of angles of different magnitudes, yet they were not sufficient to enable the children to appreciate the teacher's words about size of angles. In other words, the physical representation was not enough, even when combined with the teacher's words, for children to understand the meaning of different sizes of angles. First the children had to focus on or disembed the angle from the rest of the shape and then they had to construct their own meanings. Suggestions made by the teacher encouraged students not only to observe but also to check equivalence and to compare angles by overlaying them. For example, Jodie and James in Year 2, on being given the tangram set, immediately checked the points of the pieces in the same way as the teacher had done during the pretest. (Victor the third member of their group was away sick during this first session.)

In the session which will now be described, both Victor and James had to test their ideas of the size of an angle before they could clarify their understandings. The confusion lay in the use of the words *large*, *middle-sized*, and *small* to refer to the size of the pieces, the size of the side lengths, and the size of angles on the pieces. Victor was also completing the task set the other children in his absence, namely making the larger triangle with the other pieces. (The other children had apparently described the task to him).

2.01 Victor has gathered together some pieces, as if matching all the small angles.

2.02 James has picked up the small triangle for the small angle and points to its right angle. . . . The teacher asks for the middle-sized angle, he picks up the middle-sized triangle.

2.03 Victor picks up the parallelogram for the middle-sized angle, discards it, and then chooses the middle-sized triangle for the middle-sized angle. He puts his finger and thumb around the right angle and says "This is the middle angle." James watches.

2.04 Jodie is quite clear about which angle is smallest and which is middle-sized and she has been drawing both the small and middle-sized angles in the book.

2.05 James matches several angles against the drawing of the small angle.

2.06 Meanwhile Victor picks up the parallelogram and runs his finger along the long side. "This is the biggest" but he places it on the large triangle together with the two small triangles to cover the large triangle. He picks up the large triangle as if showing the small angle, "This is the biggest."

2.07 Jodie takes it and says "No, it isn't," and places it on the drawing of the small angle. The teacher confirms, "That's the small angle."

2.08 Meanwhile Victor has picked up the parallelogram, tests it against the drawings of the small and middle-sized angles and draws the large angle into the book.

2.09 He goes back and draws along the whole length of the arms.

2.10 The teacher suggests they make points (angles) by joining smaller points together. Victor puts points on top of each other.

2.11 James puts two small points together. The teacher says he has made the middle-sized one, praises him, and suggests that he can draw it in their book. James nods his head but he doesn't look convinced that he knows what he did. He puts two more together on top of the middle-sized point of the large triangle.

2.12 Meanwhile Victor has been trying to cover the large triangle, and he can see how to make it with the square and two triangles.

2.13 Jodie has now made the big angle with the angles of the two small triangles. (Figure 1.)

The discussion indicated how Victor, who seemed to know what was meant by the size of points (the word generally used by these children to refer to *angle*), temporarily considered that he should be comparing the size of the sides of the shapes (paragraphs 2.06 and 2.09). The interaction between students helped Victor to clarify what was meant by "the point of the same size" (paragraph 2.07). James, who had been able to match points in the first activity, began this later session by choosing the wrong points, largely because he was choosing the small or middle-sized triangles (paragraph 2.02). He soon established the meaning by listening to the teacher and to Jodie (paragraph 2.04) and by checking points with the drawing (paragraph 2.05). Later, although he successfully joined points and the teacher talked about joining angles to make new ones, he was only sure that he had the correct idea when the teacher encouraged him to show his new angle and praised him for his work (paragraph 2.10). Through manipulations, Victor (paragraph 2.12) spent some time relating the operational concept of making angles by joining together two angles to the making of congruent shapes.

Later the group made shape outlines and Victor explained that James had not made a right-angled triangle as James had thought but that he had just made an equilateral triangle in another orientation. Victor himself had made the right-angled isosceles triangle with the long side horizontal and he checked it with the tangram piece which he put on top ("a lid," he called it). The teacher asked the children what was meant by bigger. Jodie replied more spread out and picked up the tangram right-angled triangle and the pattern-block equilateral triangle, put one on top of the other and said, "See it is bigger."

The expectation of the students was to decide on the different sizes of the angles and they focused their attention on this responding by manipulating and discussing. With the later experiences there were still clarifications to be made especially when Victor was intent on completing the previous activity for which he was absent. When asked about his answers to the questions on angles in the posttest (the shapes now represented on paper), he had not initially noted the right-angle on a triangle in the same turned position which he had made with matchsticks and sticks. Later, when the page was turned for him, he said he could now see it was the same shape so it had to be the same angle. For another question, he noted that two right-angles were the same, "It is blunt like this one." His concepts of equality of angles involved the congruency of shapes, an informal description of the angles, visual imagery of the angle in different orientations, representations of angles by opening fingers and using sticks, an angle as the joining of smaller ones, and use of the word, for example, *large* to refer to different aspects of a triangle (an angle, overall size, length of side).

Students frequently noticed the equality of angles more that that of sides. The tendency to notice angles but not sides was due to holistic imagery and to an inability to disembed sides from the rest of the shape (Owens, 1993b; 1994; Owens & Clements, in press). The following is an extract from Year 2 children working next to each other with their own tangram sets.

3.01 Jonah makes a large parallelogram from the two large triangles. Sam says it is like the parallelogram piece which he picks up.

3.02 Lois makes her own parallelogram at a distance and watches as Sam matches the various angles of the parallelograms; she does likewise.

Informally Sam was showing similarity meant same size angles; he was recognising and making use of his understanding of size of angle. Students generally recognised angles on pieces which were likely to fit an angle to complete a jigsaw shape.

Although angles were perceptually strong, they were often difficult to describe. Students often said they were corners (right angles) or sharp. "The sharper it is the bigger it is" thought some, but Jodie, in fact, deliberately took on the opposite word to give the correct sense of size and stated that "the flatter it is the bigger it is,". Students commented on the spread of arms of an angle, represented by thumb and fore-finger, to explain why two angles were not equal. The teacher's interaction with the Year 2 students sparked a high degree of understanding of the size of angles and overall shapes. In the groups of three, who were asked to explain their thinking for every session, all the children (including the Year 2 children and the lower ability children) grasped the comparison and size of angles, and made other angles. In the classroom situation, fewer children were really sure about what they were doing, suggesting that not only did they have difficulties in disembedding the angles from the shape but also that student-teacher interactions assisted learning of the angle concept. This interaction was one of the influences on students' cognitive processing, especially on their selective attention. Words also gave students a means of marking aspects of their manipulations and what they were noticing.

Aspects of Problem-Solving

Responsiveness

Each of the above extracts illustrate the importance of students' responsiveness during active engagement in problem-solving activities, precipitated by their own thinking and feeling. It is a mental action for a specific context; a part of an interaction. It is manifest in physical responses such as expressions, movements, and words which, in turn, influence the context. Responsiveness is the movement forward, the risk-taking of problem-solving. Often multiple thoughts have to be held for consideration and action over several seconds or minutes until the context reacts to the development. The context change might be a verbal reply from a friend or the new position of materials when acted upon. Responsiveness implies a degree of understanding of the situation as well as involvement and interest in the activity (Owens, 1994).

Responsiveness results from a combination of cognitive processes which include attending, perceiving, listening, looking, visual imagining, conceptualising, intuitive thinking, and heuristic processing (such as establishing the meaning of the problem, developing tactics, self-monitoring and checking). Cognitive processing also incorporates affective processes such as reactions to the organisation of the classroom and to success, confidence, interest, and tolerance of open-ended situations. Problem-solving episodes or points involving critical change in thinking are likely to involve both changes in affect and changes in understanding.

Selective Attention

One particular aspect of cognitive processing, namely selective attention, gradually emerged from the study as extremely important in problem solving. In each of the above episodes, the students attended to particular aspects that affected their understanding of an angle. A student's attention may have been focused, for example, on the angle, the whole piece, its shape, the sides, a comment of a friend or the teacher, the result of an action, or their own thoughts and imagery. Turning fingers apart was used by students to focus attention on the size of static angles of shapes because turning is operational and hence a means by which children kinaesthetically appreciated and visualised the concept.

Selective attention initially links long-term memory with sensed perceptions, and then relates the perception to what is stored in memory. Selective attention guides a student's responsiveness to a problem and leads students to choose between acts. Actions are performed one at a time although several features can be perceived at the same time (Kaufmann, 1979). In the study, imagery was an important aspect of problem solving because it often (a) guided students' responsiveness, (b) assisted them to maintain a holistic understanding, (c) enabled them to react intuitively, (d) focused their attention, (e) assisted them to monitor their progress, and (f) helped them shift to another approach by attending to another aspect of the image.

Imagery was supported by conceptualisation which, in turn, was aided by comments from the teacher or from other students, and by observations of other students' work. As a result, students manipulated the materials in certain ways. After observing the results and making decisions about the suitability of certain actions, further manipulations were made. Other interactions with materials such as matching pieces or parts of pieces to compare the size of angles assisted the development of the concept of angle and of size of angles. Students were able to test their concepts against the materials and in this way they were able to assess their own progress. They frequently turned the fore-finger away from the thumb in order to consider angle size.

Conclusion

This study has shown that learning about angles is enhanced through problem solving situations that encourage students to focus on angle size. Students check their developing concepts by interacting with peers or the teacher or by manipulating materials. Students are being responsive and engaging in learning, noticing and attending. Some intention is gained from the words of others. Students analyse and note size as part of their early recognition of angles although it is not initially a stable concept. Further focusing, checking, and experiencing of different orientations and contexts assists the concepts to develop. Without responsiveness, the students would not be acting and constructing meaning. The meaning of angle is embedded in numerous conceptual frameworks which develop concurrently and integrally with visual imagery.

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Is there a need for a *Mathematics Intervention* program in Grades 3 and 4?

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The standard of numeracy skills of young Australians continues to be of concern to mathematics educators, teachers, parents and politicians. This paper discusses whether a mathematics intervention program developed for Grade 1 warrants extension into Grades 3 and 4. The focus is on 1997 data that show that although Grade 3 and 4 students had made significant progress since Grade 1, there is still need for additional assistance in Grades 3 and 4.

Introduction

Considerable concern is currently being expressed by mathematics educators, teachers, parents and politicians, about the standard of the literacy and numeracy skills of young Australians. Recently Australian Commonwealth, State and Territory education ministers agreed that "every child leaving primary school should be numerate, and be able to read, write and spell at an appropriate level" (Masters & Forster, 1997). They further agreed that "every child commencing school from 1998 will achieve a minimum acceptable literacy and numeracy standard within four years" (McLean, 1997, p.1). In this context numeracy was defined as "the effective use of mathematics to meet the general demands of life at school and home, in paid work, and for participation in community and civic life" (McLean, 1997, p. 2).

A National Plan (Kemp, 1997) has been developed to ensure that these goals are met. The plan requires that education authorities provide support for teachers in their task of identifying children who are not achieving adequate literacy and numeracy skills and in providing early intervention strategies for these students. *Mathematics Intervention* is one such program that was designed, implemented and monitored by classroom teachers in the hope that their intervention strategies would assist children achieve adequate numeracy skills (Pearn & Hunting, 1995; Pearn & Merrifield, 1996).

Mathematics Intervention

Mathematics Intervention was developed as a collaborative project involving the principal and staff of a state primary school in the metropolitan area of Melbourne and mathematics educators from a nearby university (Pearn & Hunting, 1995; Pearn & Merrifield, 1996). The program, first implemented in 1993, aims to identify, then assist, children in Grade 1 "at risk" of not coping with the mathematics curriculum, as suggested in the National Statement on Mathematics for Australian Schools (Australian Education Council, 1991).

In November 1997 testing was extended into Grades 3 and 4. At that time 57 students were tested. Of these, 32 had been tested when they were in Grade 1. There were two reasons for the 1997 testing program. The first was to validate the effectiveness of the Grade 1 *Mathematics Intervention* program currently operating at the school. The second was to determine whether there was a need to establish a similar program for Grade 3 and Grade 4 students. This paper focuses on the second of these aims, that is, to determine whether there is a need to establish a *Mathematics Intervention* program for students in Grades 3 and 4. To investigate this need a procedure similar to that introduced when developing the Grade 1 *Mathematics Intervention* program was used. As it was anticipated that the Grade 3 and 4 *Mathematics Intervention* program would be an extension of the same underlying mathematics research the rationale and scope of the Grade 1 *Mathematics Intervention* program is documented.

Previous research

Since its introduction in 1993, the Grade 1 *Mathematics Intervention* program has featured elements of both Reading Recovery (Clay, 1987) and Mathematics Recovery (Wright, 1991; 1996). *Mathematics Intervention* offers students the chance to experience success in mathematics by developing the basic concepts of number upon which they build their understanding of mathematics. Students are withdrawn from their classes and work in small groups with a specialist teacher to assist with the development of their mathematical skills and strategies.

The theoretical framework underpinning the Grade 1 *Mathematics Intervention* program is based on recent research about children's early arithmetical learning (Steffe, von Glasersfeld, Richards and Cobb, 1983; 1988; Wright, 1991; 1996) and about the types of strategies used by children to demonstrate their mathematical knowledge (Gray & Tall, 1994). In particular, the program aims to assist students' progression through the counting stages as developed by Steffe et al. (1983, 1988) and which are summarised below. It was planned that the Grade 3 and 4 *Mathematics Intervention* program would similarly use this same theoretical framework.

Counting stages: Research into *Mathematics Intervention* has confirmed the research by Steffe et al. (1983; 1988) that stated children move through the counting stages documented below.

1. *Perceptual*. Students are limited to counting those items they can perceive.
2. *Figurative*. Students count from one when solving addition problems with screened collections. They appear to visualise the items and all movements are important. (Often typified by the hand waving over hidden objects.) If required to add two collections, one of six items and one of three items, the student must first count the six items to understand the meaning of "six", then count the three items, then count the whole collection of six and three.
3. *Initial number sequence*. Students can now count on to solve addition and missing addend problems with screened collections. They no longer count from one but begin from the appropriate number. If adding collections, one of six items and one of three items, students commence the count at six and then count on: six, seven, eight, nine.
4. *Implicitly nested number sequence*. Students are able to focus on the collection of unit items as one thing, as well as the abstract unit items. They can count-on and count-down, choosing the most appropriate to solve problems. They generally count down to solve subtraction problems.
5. *Explicitly nested number sequence*. Students are simultaneously aware of two number sequences and can disembed smaller composite units from the composite unit that contains it, and then compare them. They understand that addition and subtraction are inverse operations.

As well as assisting students to progress through the counting stages the *Mathematics Intervention* program focuses on the strategies used by students when solving mathematical tasks. In particular, the focus is on work by Gray and Tall (1994).

Strategy choice: Assessment for the Grade 1 *Mathematics Intervention* program has confirmed research studies by Gray and Tall (1994) that have shown that young children who are successful with mathematics use different types of strategies from those who are struggling with mathematics. Students struggling with mathematics are usually procedural thinkers, dependent on the procedure of counting and limited to the "count-all" and "count-back" procedures. In summary, Gray and Tall (1994) defined procedural thinking as being demonstrated when:

> ... the numbers are used only as concrete entities to be manipulated through a counting process. The emphasis on the procedure reduces the focus on the relationship between input and output, often leading to idiosyncratic extensions of the counting procedure that may not generalize (p. 132).

For example, when asked to count back from a given number, students have been heard to count up to each number before responding with the number required. This procedure is highly unlikely to generalise into a backwards counting sequence.

While some students tested in Grade 1 were dependent on rules and procedures, other students gave instantaneous answers. For example, when students who gave an instant correct response to tasks were asked "How did you do that" they gave several different strategies they could have used and then checked that their solutions were correct. According to Gray and Tall (1994) this use of known facts and procedures to solve problems, along with the demonstration of a combination of conceptual thinking and procedural thinking, indicates that these children are proceptual thinkers. Gray and Tall (1994) defined proceptual thinking as:

> ... the flexible facility to ... enable(s) a symbol to be maintained in short-term memory in a compact form for mental manipulation or to trigger a sequence of actions in time to carry out a mental process. It includes both concepts to know and processes to do (pp. 124-125).

Another feature of the Grade 1 *Mathematics Intervention* program requires teachers to use a one on one clinical interview to assess the extent of the child's mathematical knowledge. This allows the teacher to observe and interpret the child's actions as he/she works on a set task. Leder (1990) advocates encouraging students to talk about their mathematical strategies to obtain information on children's own mathematical understanding and knowledge.

> Encouraging students -- particularly those deemed to have difficulties -- to talk about mathematics and listening carefully to what is being said provides invaluable information about students' learning. It is a strategy that can be used readily by classroom teachers to probe and monitor their students' learning. The data obtained can serve if necessary, as a rich data base for subsequent error analysis (p.26).

Grade 1 Initial Assessment

In 1997, children who had been tested in 1994 and 1995 when in Grade 1, were now in Grade 4 and Grade 3 respectively. As we are going to look at the progress of these children later in the paper, Table 1 shows the results of the testing not only from 1997, but also from 1994 and 1995. The results of the small group of 32 students who were tested both in Grade 1 and in 1997 are given in bold in Table 1. The results in the brackets are those of the whole year level.

Table 1: Results from Level AA tests (Grade 1): 1994, 1995 and 1997 (in percentages).

Year	ones	back 20-1	back 10-1	twos	fives	tens	6 - 12	before /after	14 beads	pattern	numeral	6+3=	10= +2
1994													
n=18	**94**	**50**	**89**	**44**	**44**	**61**	**61**	---	**83**	--	**67**	---	**72**
(n =48)	*(98)	(69)	(77)	(44)	(48)	(67)	(69)		(88)		(71)		(71)
1995													
n=14	**100**	**64**	**93**	**71**	**50**	**57**	**86**	**86**	**100**	**100**	**43**	**93**	**64**
(n=26)	(96)	(65)	(96)	(50)	(50)	(54)	(88)	(92)	(100)	(96)	(46)	(85)	(69)
1997													
(n=62)	100	73	97	53	47	66	63	84	90	94	37	79	56

* Figures in brackets are the results of testing all students at this level and in this year.

Of the 13 tasks used in the initial Grade 1 interviews in 1994 there is good overlap in the results for both the small group and the whole cohort of students for all but two tasks: counting backwards from 20 and counting backwards from 10. The small group achieved better results for counting backwards from 10 but were less successful in counting back from 20 than the whole cohort.

In 1995, the small group's results overlapped with the whole cohort results in all but three of the tasks: they achieved better results as a group on counting by twos and the first counting stage task but were less successful on determining the number before and /or after a given number.

Table 1 highlights that the results for the small group are representative of the Grade 1 results for the whole group. Most Grade 1 children, over the three years given, were successful in counting forwards by ones to 20 and backwards by ones from ten, counted patterns of dots and counted out exactly 14 beads. They were less successful identifying the numbers between the numbers six and twelve or determining numbers "before" or "after" a given number. Children included in *Mathematics Intervention* were those who displayed difficulties with most tasks and were at or below Stage 1 and used procedural strategies such as "count all".

Extending *Mathematics Intervention* into Grades 3 and 4

The 1997 Grades 3 and 4 interview tasks were designed by two of the teacher-clinicians involved in designing the original set of tasks for the Grade 1 clinical interview. It was decided to maintain the focus on number as initiated in the Grade 1 *Mathematics Intervention* program although we acknowledge the importance of a breadth of mathematical activities. However, as the Curriculum and Standards Framework [CSF] states:

> As a student acquires an appreciation of different levels of understanding of number, intersections occur with other mathematical studies in ways which give number a central unifying role. Work in the number strand links with work in all other strands ... Later work in all strands requires that they understand and work confidently with all kinds of numbers (Board of Studies, 1995, p. 42).

To determine each child's development in terms of the counting stages, and the types of strategies they used in solving mathematical tasks, the interview included tasks testing knowledge of verbal counting sequences, whole numbers, oral responses to addition and subtraction facts and word problems. Whole number tasks were given orally by the teacher while cards showing the task were also presented simultaneously. For example, for the subtraction task *52 -17* the child was asked "Can you tell me what 52 take away 17 is?" and at the same time a card was presented on which 52 -*17* was written horizontally. As with the Grade 1 interviews the Grade 3 and 4 tasks were designed so children were given every opportunity to demonstrate the various strategies they used to solve mathematical tasks. This interview was trialed in November 1997 and is now under review.

Grade 3 and 4 results from clinical interviews

Most Grade 3 and 4 students were able to count by ones, fives and tens but several students had difficulty counting forwards by twos starting at 15 and backwards by tens starting at 120 although most students could successfully count backwards by tens from 100. It appeared that they had learnt some sequences by rote but had no strategies for counting from a specific number.

Most students were successful in identifying the number 'one more than', 'ten more than', and '100 less than' 342. They found the missing addend and subtraction tasks much more difficult due to an apparent reliance on rules and procedures. For example, six students from the same class all responded that 52 - 17 was 45 because "you take the small number from the large one". Difficulties experienced with subtraction are reflected in Table 2 where results from computational tasks are given. Although most students were able to identify the word problem as subtraction, 15 of the group of 30 were unable to complete the computation successfully due to a reliance on faulty or "buggy" algorithms.

Table 2: Results of 1997 Grade 3 and 4 computational skill tasks (in percentages).

Year Level	addition facts	subtraction facts	addition problem	subtraction problem	multiplication problem	division problem
Grade 3 n = 13 (24)*	**86** (71)*	**50** (54)	**86** (88)	**50** (42)	**57** (63)	**64** (63)
Grade 4 n = 17 (33)	**72** (70)	**61** (61)	**89** (91)	**56** (52)	**56** (55)	**72** (79)
Total n = 30 (57)	**78** (70)	**59** (58)	**86** (89)	**53** (47)	**56** (58)	**69** (72)

* Figures in brackets are the results of Grades 3 &4 interviews of 57 students.

To illustrate the additional information revealed in using a clinical interview rather than a paper and pencil test, Table 3 includes three responses given by students in response to the subtraction word problem. Students were given the problem which they were asked to read aloud. If they experienced difficulty in reading the problem it was read for them.

Table 3: Typical responses to the subtraction word problem.

Task	Lynda	Mike	Barry
Richard is 131 cms tall. Mary is 17 cms shorter than Richard. How tall is Mary?	Successfully counted back by ones from 131 keeping track on her fingers. Gave correct answer	Took 10 away from 31 then 7 away from 21. Gave correct answer.	Written algorithm. 131 - 17 126 Incorrect answer.

It can be seen that Lynda used a procedure of counting back by ones. Mike successfully used an 'invented' strategy while Barry used a 'buggy' or faulty procedure which he used consistently through the interview for all subtraction tasks.

A preliminary investigation of the Grade 3 and 4 results has highlighted extremes in the knowledge and strategies used. Those who appeared to be struggling with mathematics relied on their memory of rules and procedures. While two of these students had participated in the Grade 1 *Mathematics Intervention* program, there were many students who had not had the opportunity to do so because they had come into the school after Grade 1.

Grade 3 and 4 students who were successful with the mathematical tasks were flexible in their use of strategies (Gray and Tall, 1994) and used appropriate "invented strategies" (Carpenter et al., 1998) when tasks were presented without access to paper and pencil. Carpenter et al. (1998) noted that many students constructed their own procedures for adding and subtracting multi-digit numbers without using physical materials. They called these strategies "invented strategies". They found that students who use invented strategies developed a knowledge of base-ten number concepts earlier than students who relied more on standard algorithms. In their study they found that children made "relatively few conceptual errors in using invented strategies, whereas children exhibited a number of buggy procedures in using standard algorithms " (p. 19).

Scoring of interview results

In Figure 1 the two sets of interview results were compared for the small group of 32 students who were tested when in Grade 1 and again in 1997. Since only two students had participated in the Grade 1 *Mathematics Intervention* program the data were given for all 32 students who had participated in Grade 1 and 1997 testing programs. To compare these results, solutions to tasks were given a numerical score, based on the knowledge displayed and the types of strategies used. For example, for the counting task in the Grade 1 interview, when asked to count backwards by ones from 20, children would score 0 if no attempt was made to count backwards, a score of 1 if successful counting backwards from 10, and a score of 2 if successful counting backwards from 20. The maximum score for the Grade 1 test was 24 and for the Grade 3 and 4 test was 54. In order to compare these results only scores for tasks common to both Grade 1 interviews in 1994 and 1995 are included. All scores have been converted to percentages.

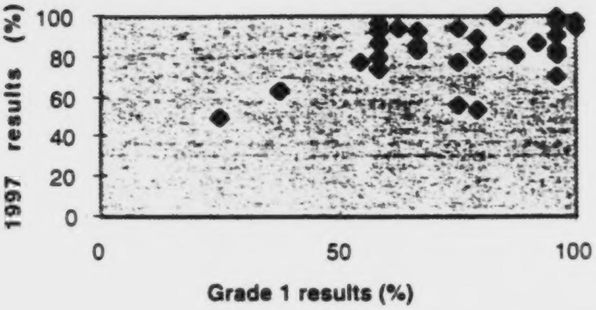

Figure 1: Comparison of results from 1994/1995 with those of 1997 (n=30).

The median for the Grade 1 results was 18.5 (77%) and the lower quartile score was 14 (58%) while the median for the 1997 results was 46.5 (86%) and lower quartile score was 42 (78%). From Figure 1 it can be noted that there are students whose results were:

- in the lower quartile in both Grade 1 and 1997,
- in the lower quartile of the 1997 results but not in the lower quartile in Grade 1,
- not in the lower quartile in Grade 1 or in 1997 but who appeared to be performing at a comparatively lower level in 1997 than predicted from the Grade 1 results.

In Table 4 the focus is on the number of children whose results place them on, below or above the 25th percentile of the 1997 results which are categorised in relation to their Grade 1 results.

Table 4: Number of children in lower quartile in 1997 and in 1994/1995 (n=32).

	1997 < 78%	1997 = 78%	1997 > 78%	Total
Grade 1 < 58%	3	1	0	4 (13%)
Grade 1 = 58%	1	0	4	5 (15%)
Grade 1 > 58%	3	1	19	23 (72%)
Total	7(22%)	2 (6%)	23 (72%)	32(100%)

Although these results must be treated with caution as this is a small sample of 32, the group is representative of the results for the larger group of 278 Grade 1 children interviewed over the last six years. The results in Table 4 indicate there were three students with results in the lower quartile in both Grade 1 and 1997. There was also one student whose 1997 score was in the lower quartile in 1997 but whose score was on the 25th percentile in Grade 1 and one student whose result was on the 25th percentile for 1997 but in the lower quartile for Grade 1. Thus there were five students ranked on the 25th percentile or in the lower quartile for both sets of results. That is, 15% of students tested had not improved their class ranking in mathematics testing over since being tested in Grade 1. There were also students whose results in 1997 appear to be less than their Grade 1 results indicated. There were three students who were above the 25th percentile in Grade 1 but in 1997 were in the lower quartile. There was also one student whose Grade 1 results were higher than the lower quartile but in 1997 was on the 25th percentile. This means that of the nine students on or below the 25th percentile in 1997, four were on or above the 25th percentile in earlier testing.

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This highlights a need for some additional assistance for two groups of students tested in Grades 3 and 4 in 1997. That is, those who have not improved their ranking since the Grade 1 testing, and those that appear to have been less successful than their Grade 1 results would indicate. Further analysis will attempt to determine reasons for this lack of success.

Discussion

Of concern to the researcher is the small group of students who were found to be in the lower quartile when tested in 1997. Of most concern is the small group of students who do not appear to have succeeded to the extent that their Grade 1 results indicated. The numbers in the sample are too small to be considered representative but two of the three boys concerned were in Grade 4, the other in Grade 3. This means that the two boys in Grade 4 were performing at a lower level than all but 4 of their peers from both Grade 3 and 4.

This research has highlighted the differences in children's mathematical knowledge and the type of whole number strategies they use when solving tasks set in different contexts. Testing in Grades 3 and 4 revealed that all children had improved considerably in their mathematical knowledge, and were using more sophisticated strategies than they had used in Grade 1. Preliminary results from the Grades 3 and 4 testing have shown that children who are successful with mathematics at this level are flexible in their mathematical thinking and use a variety of strategies, including invented strategies. However, there were Grade 3 and 4 students who were still experiencing difficulties with the verbal counting sequences. There were several students who used the 'count-back' and the 'count-on' strategies while some used 'invented' strategies (see for example Mike's response in Table 3). However, children like Barry (see Table 3), struggling with mathematics, rely on rules and procedures even when these are inefficient and unreliable. Hiebert and Lefevre (1986) have highlighted the difficulties of relying on rules and procedures:

> Procedures that lack connections with conceptual knowledge may deteriorate quickly and are not reconstructable; they may only be partially remembered and combined with other subprocedures in inappropriate ways; they often are bound to the specific context in which they are learned and do not transfer easily to new situations; and they can be applied inappropriately without the benefit of a validating critic to check the reasonableness of the outcome (pp. 21-22).

Conclusion

It would appear from our preliminary study that Grade 3 and 4 students "at risk" need special assistance to succeed at this level. Although all students had showed improvement in their mathematical knowledge and the types of strategies they used since the Grade 1 testing some children had improved to a lesser extent than their peers. There were also students who were not achieving at the level predicted by their Grade 1 results.

For students "at risk" of not succeeding at mathematics, one of the greatest difficulties is that they do not possess either rich mathematical ideas or the powerful strategies that will enable them to use their mathematical knowledge to improve and enhance their mathematical thinking. A *Mathematics Intervention* program at Grade 3 and 4 would not only need to determine the child's counting stage as for the Grade 1 program, but would also need to concentrate on developing more efficient strategies for computational tasks. All students need to develop strategies to enhance their understanding of number concepts rather than being left dependent on rules and procedures which make no sense to them. Students should be encouraged to share their own "invented strategies" with each other and use and discuss alternative strategies for solving mathematical problems in the social context of their classroom.

This research program is continuing. The need for additional assistance and provision of special programs for students "at risk' with the 1998 testing of Grade 3 students, who were tested in Grade 1 in 1996, is being examined. The challenge is now to develop a *Mathematics Intervention* program for Grade 3 and Grade 4 students based on mathematics education research. This program should include the research of Steffe et al. (1993, 1988) into the five counting stages and that of Gray and Tall (1994) into strategy choice. The research of Carpenter et al. (1998), which highlights the need for students to develop flexibility in their mathematical thinking and a range of strategies so they become less reliant on rules and procedures, should also be included.

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Year 3 students' place-value misconceptions: Another look at MAB

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Sixteen Year 3 students were interviewed individually about two- and three-digit numeration concepts, before and after ten groupwork sessions. One group of questions asked students about values represented by each digit in a two-digit number. Responses revealed that some students had not developed the idea that the "tens" digit represents a collection of ones, but believed that it represented only its face value. Comparison of results before and after the groupwork sessions showed that a number of participants had improved understandings of values represented by individual digits at the conclusion of the study. These results have implications for how place-value concepts are developed with place-value blocks.

Introduction

The author has previously reported on a study in progress, in which Year 3 students' understanding of place-value concepts is being evaluated (Price, 1997). The central objective of the study is to investigate differences in learning that result from use of either conventional place-value blocks or computer software. Participants were placed in groups of four students, and each group was involved in ten daily groupwork sessions, in which they used place-value material (blocks or a computer) to model numbers to answer questions about two-digit and three-digit numbers. This paper focuses on results from two interviews given individually to each participant, before and after groupwork sessions.

The Base-ten Numeration System

The Hindu-Arabic numeration system is a place-value system with a base of 10; each individual digit in a multidigit number represents a value equal to the product of its face value and the appropriate power of ten. It enables numbers of virtually any size to be represented unambiguously and succinctly, and is an essential component of many fields, including finance, industry and science. Numbers recorded in the base-ten numeration system are also used heavily in other school curriculum areas, including science, economics and geography. Thus learning place-value concepts is an important foundation for students' future needs, both within and beyond their schooling.

It is appropriate to consider the concepts that underlie the base-ten numeration system that are regarded as essential for students to understand by the end of primary school, and how they can be taught. These concepts can usefully be grouped into three types: basic number concepts, concepts of abstraction and system concepts.

Basic number concepts relate to single digit numbers, and are prerequisites to developing an understanding of the base-ten numeration system. These include knowledge of the "face values" of the number digits 0 to 9, the use of symbols to represent quantities, and basic counting concepts and skills. Each of these understandings is necessary before a child can begin to learn about multidigit numbers.

Concepts of abstraction relate to basic number concepts, and are also prerequisites for grasping multidigit number concepts. At the heart of the use of numerical symbols (whether written, pictorial or physical) is a central abstraction—that numbers can be represented or symbolised, but can never be perceived directly with the physical senses. Abstract concepts include the concept of number and the twin concepts that symbols represent numbers in written communication and that material objects, such as counters and place-value blocks, may represent quantities by illustrating their numerosity.

Third, *system concepts* relate to the numeration system itself and the rules that govern how it is used to represent quantities. System concepts include the ideas that unit values represented in each successive place map onto successive powers of ten, that the value of a place is determined by reference to the right-most whole-number place and that the value of each digit equals the product of the digit's face value and the appropriate power of ten.

Primary mathematics curricula typically include the development of understanding of these many concepts as a major objective. Benchmarks for inclusion of concepts at various year levels provide for development of the concepts described above over the entire span of primary year levels. Basic number concepts are developed in the first year of schooling, as young students are introduced to the principle of cardination, to the skills of counting a finite number of objects fewer than 10 and to the symbols that represent the numbers 0 to 9. In the succeeding years, students are introduced to the numeration system, starting with two-digit numbers and building up to very large numbers. For example, in Queensland two-digit numbers are introduced in Year 2, three-digit numbers in Year 3, four-digit numbers in Year 4 and five-digit numbers and beyond in Year 5. Teaching about numeration of decimal fractions similarly progresses year by year.

Use of schema in understanding numeration concepts

Cognitive science has provided researchers and educators with a useful notion, the *schema*, which can help to understand what is necessary for a student to develop understanding of a complex concept such as numeration. Schemas have been defined variously by different authors; Derry (1996) noted that "*schema* is a general term connoting virtually any memory structure" (p. 167). Because numeration concepts described earlier are integrated into a complex whole, a schema is needed to relate the various components of the system to each other. Thus for a student to gain a deep understanding of the base-ten numeration system, he or she needs to develop a schema to organise the thinking involved in using the system to solve mathematical problems. Such a schema should enable the student to see connections between aspects of numbers and their uses in novel situations.

The idea of assisting students to develop powerful schemas to understand the complexities of the base-ten numeration system is very attractive to mathematics educators. However, anecdotal and research evidence of the understandings of students reveals that for many students such organised mental structures for multidigit numbers have not developed. Rather, many students seem to have little idea of the connections among numbers and number ideas, and instead rely on memorised procedural rules to answer mathematical questions (e.g., Fuson & Briars, 1990). There is therefore a pressing challenge for mathematics teachers to teach in a way that helps students to develop understandings that can be applied to new situations, based on understandings of the structures inherent in the numeration system. The research in progress described in this paper is showing that use of place-value blocks by young students does not necessarily result in the development of powerful models of the base-ten numeration system, and that teachers need to monitor carefully how such materials are used by themselves and by students.

Modelling numbers with concrete materials

In attempting to help students to develop accurate, flexible understandings of the numeration system, primary teachers use a variety of concrete materials in their teaching of mathematics. The extent of this use is indicated by work conducted by Gilbert and Bush (1988) in the US, and by Howard, Perry and Conroy (1995) in NSW. Both these studies showed frequent use of concrete materials by the surveyed teachers. Gilbert and Bush found that 65% of respondents reported using concrete materials at least once per week; in Howard et al.'s study 62% of respondents reported using concrete materials "often", with only 2 teachers out of 249 reporting that they did not use them at all.

Place-value blocks as analogues of numbers

One material that needs to be singled out in the context of helping students to develop number understandings is place-value blocks (known in Queensland as Multibase Arithmetic Blocks [MAB]). As pointed out by English and Halford (1995), the structure of place-value blocks mirrors the structure of the base-ten numeration system analogically, with the size of each block being proportional to the value it represents, compared to the unit cube.

Place-value blocks have been recommended for teaching numeration concepts since they were developed in the 1960s, and are "probably the most commonly used analogs in the teaching of numeration and computation" (English & Halford, 1995, p. 105). This statement is supported by Howard et al. (1995), who reported that the material most commonly used by their respondents was "base 10 material", used by 84% of respondents. Howard et al. noted that this frequency of use was probably related to two factors. First, 96% of teachers responding to the survey stated that they believed that concrete materials "benefit children's learning". Second, "the syllabus and many commercial mathematics programs encourage the use of such material" (p. 6).

Research results

This confidence in place-value blocks' effectiveness for teaching numeration concepts, expressed by both academics and classroom teachers, has to be tempered by other comments, however. Research into the effects of instructional use of place-value blocks has not produced the positive, unequivocal results that were hoped for. Hunting and Lamon (1995) suggested that many variables may affect results from use of materials, including the type of material, length of time used and teacher training. One fact that needs to be considered by teachers wishing to use concrete materials, pointed out by Hunting and Lamon (1995), is that the mathematical structure of multidigit numbers is not contained in any material; the structure of the numbers themselves has to be constructed in the mind of the individual student. Thus there is no guarantee that use of physical materials will lead to better understanding of numbers. Baroody (1989) added support to this point with his statement that learning can only take place when the "[learning] experience is meaningful to pupils and . . . they are actively engaged in thinking about it" (p. 5).

Thompson (1994) suggested that research studies needed to give attention to the broad picture of the teaching/learning environment to discern reasons for the success or failure of instructional use of concrete materials:

> These contradictions [among findings of different studies] are probably due to aspects of instruction and students' engagement to which studies did not attend. Evidently, just using concrete materials is not enough to guarantee success. We must look at the total instructional environment to understand effective use of concrete materials—especially teachers' images of what they intend to teach and students' images of the activities in which they are asked to engage. (p. 556)

The study reported on here used a descriptive approach to data gathering, relying on videotapes of groupwork sessions to reveal important aspects of the learning environment and its effects on students' learning.

The Research

The research study reported on here investigates Year 3 students' understandings of multidigit numbers. Part of the study involved individual interviews of the participants, in which the student was asked a series of questions probing their understanding of two- and three-digit numbers. The students' responses indicated important mis-conceptions about two-digit numbers held by several of them.

The main part of the study comprised ten consecutive groupwork sessions, in which groups of four students worked on a series of questions on cards, designed to revise two-digit number concepts, and to teach initial three-digit number concepts. Two different "treatments" were used, with half the groups using conventional place-value blocks, and half using computer software, to help them answer the questions.

Participants

Participants in the study reported here were selected from the population of Year 3 students (aged 7-8 years) at a primary school in a small Queensland rural town. Participants were selected at random from two groups: students of either high or low mathematical achievement, based on the previous year's *Year Two Diagnostic Net*, a state-wide test used in Queensland to identify students at risk in the areas of literacy and numeracy. Students were assigned to groups that were matched for ability, on the assumptions that maximum learning would be possible if children in each group were of similar mathematical ability. Each group comprised two girls and two boys. Four groups were used, with one high- and one low-achievement level group using each of the blocks or the computer.

Interview questions

The particular interview questions reported on here were used to ask each student about the values represented by the digits in a two-digit number. Similar "digit correspondence" questions have previously been used by Ross (1989) and Miura, Okamoto, Kim, Steere and Fayol (1993). The student is asked to count a number of objects between 10 and 100, and asked about the value represented by each digit.

The two interviews in the present study included parallel questions, using different numbers. The digit correspondence questions from the first interview are shown in Figure 1. Students were asked to count 24 pop sticks, write the symbol for 24 and explain the values represented by each digit in the written symbol. In the second interview the same questions were asked, based on a group of 37 pop sticks.

- Show the student 24 pop sticks.
- Ask the student to count them (correct if necessary), and to write the symbol for the number.
- Circle the "4", and ask "Does this part of your 24 have anything to do with how many sticks you have? Please show me."
- Circle the "2", and ask "Does this part of your 24 have anything to do with how many sticks you have? Please show me."

Figure 1 Digit correspondence questions from first interview

Results

Results from the first digit correspondence question above are summarised in Table 1. Of the 16 participants, 14 knew that the "4" in "24" stood for 4 of the sticks, but only 7 could correctly state that the "2" in "24" stood for two collections of ten sticks.

Table 1 *Summary of results of digit correspondence questions (first interview)*

Question: *Does the "4" in your "24" have anything to do with the number of sticks?*	
Response	**Number of Responses**
4 sticks	14
No	2
Does the "2" in your "24" have anything to do with the number of sticks?	
Response	**Number of Responses**
20 sticks	7
2 sticks	8
No	1

All participants but two knew that the "4" in "24" represented 4 sticks. The two students who differed both said that the symbol 4 did not have a referent or referents among the sticks. One student, Kelly, said that "you don't have 2 sticks there and you don't have 4 sticks - that's what I know". She agreed that the symbol "24" represented the group of 24 sticks, but believed that the individual digits did not refer to any particular number of sticks. The other student, Simone, who claimed that the "4" had no referent appeared to have a confusion that was related to place-value blocks. Simone said that there were tens among the sticks, but no ones. She indicated 2 sticks as standing for the symbol "2", but said that a "one" would have to be smaller than a single stick. It appears that she was confusing the shape of a stick with a "ten block" in place-value blocks. Evidently she had learned to associate the long shape of the ten block, rather than its size relative to a one block, with ten. A similar confusion was observed in one of the boys, who asked if he should count the sticks as "1, 2, 3, ...", or as "10, 20, 30, ...".

The most obvious confusion held by these children, however, was that the "2" in the symbol "24" represented only its "face value". This conception of values represented by individual digits in a multidigit number has been reported widely in the literature (e.g., Bednarz & Janvier, 1982; Miura et al., 1993). Ross (1989) described it in this way:

> Students interpret each digit as representing the number indicated by its face value . . . . but these objects do not truly represent groups of ten units to students in [this stage]; students do not recognize that the number represented by the tens digit is a multiple of ten. (p. 49)

The participants in this study who revealed this misconception were asked by the interviewer to confirm that the symbol "24" represented all 24 sticks. He then asked about the remaining 18 sticks. Answers given by these children included that the other sticks were "left-overs", were "not included", were "just extras" or were "just left out". These children were asked to explain this, but none were able to do so. A few children did not attempt to explain the 18 extra sticks, but maintained two mathematically-contradictory beliefs about the symbols. They explained that the symbol "24" represented all the sticks, and also said that the "2" and "4" represented only 2 and 4 sticks respectively, but could not integrate these two conceptions into a unified idea.

Second interview results

In the second interview, after the participants had spent ten sessions working with materials modelling two- and three-digit numbers, participants' responses to digit correspondence questions differed. The students were asked to count 37 pop sticks, and to explain the "3" and the "7". Results are summarised in Table 2. There was no significant difference between students who had used blocks and those who had used the computer, so these are not reported here.

Table 2 *Summary of results of digit correspondence questions (second interview)*

Question: *Does the "7" in your "37" have anything to do with the number of sticks?*	
Response	**Number of Responses**
7 sticks	14
No	2
Does the "3" in your "37" have anything to do with the number of sticks?	
Response (initial)	**Number of Responses**
30 sticks	6
3 sticks	7
No	3
Response (final)	**Number of Responses**
30 sticks	11
3 sticks	4
No	1

The main difference between responses on the first and second interviews was that in the second interviews a number of participants changed their responses when the interviewer asked further questions to probe or challenge their initial response. Initially the number of students who showed the correct number of sticks for the symbol "3" was still low—in fact it was one less than the number who answered the equivalent question correctly in the first interview. However, as the interview progressed, an additional five students agreed that the "3" represented 30 sticks. In some cases this was due to the student changing their opinion without direct prompting from the interviewer. In other cases, students altered their answer after the interviewer asked further questions. Similar questions had been asked in the first interview, but at that time students did not alter their answers significantly. It appears, however, that in the second interview, although initially more than half of the participants were confused about the values represented, with further questioning they were able to give a correct response. This is illustrated by the following transcript excerpt:

I: *Does this part [7] of your "37" have anything to do with how many sticks you have?*

Maxine: Yes.

I: *Can you show me?*

Maxine: [puts out 7 sticks]

I: *Does this part [3] of your "37" have anything to do with how many sticks you have?*

Maxine: Yes.

I: *Can you show me?*

Maxine: [picks out 3 sticks]

I: *Yes, alright. I'll just put those there. So this 7 is for 7 like that and 3 is for 3 like that. What about those [remaining 27 sticks]?*

Maxine: [pauses, then counts out remaining sticks then puts them with the 3]

I: *[laughs] Now you are saying that that 3 is for these?*

Maxine: [nods]

I: *How many are there?*

Maxine: 30.

Maxine apparently realised that the symbol "3" stood for the remaining sticks, though initially she only used 3 sticks. At least two other students realised that there was a contradiction between the symbol "37" representing all 37 sticks, and their belief that the single digits represented only 3 and 7 sticks respectively. They resolved this difficulty by saying that the single digits did not represent any sticks individually, but only when combined into the two-digit number, as shown in the following transcript:

I: *Does this part [7] of your "37" have anything to do with how many sticks you have?*

Hayden: No.

I: *I doesn't? OK, can you tell me what that 7 means?*

Hayden: It's a part of 30 ... it's a part of like in 30 it's a part like ... you count to 30 and then you count 7 more and it ends up 37.

I: *And what does this 3 here mean?*

Hayden: It's up ... it's up to 30 ... like if you count up to 30.

I: *Uh-huh. Can I show you something? If we have 7 sticks like that [puts out 7 sticks], could we say that 7 is for 7 like that?*

Hayden: No, because that ... that's not like [picks up 3 sticks] ... that's only 11 [puts them down next to the 7 sticks].

I: *Why have you got those 3? That's for the 3 is it?*

Hayden: No, those aren't for the 3.

I: *It's not for 3? So the 7 is not for 7 either?*

Hayden: No, because it doesn't make um 37. It only makes 11.

I: *But the whole number written down like that is for all of them?*

Hayden: Yep.

I: *But if you take just the 7 it's not ... you can't take part of them and say that part is for that?*

Hayden: No.

Hayden clearly knew that the two-digit symbol represented all the sticks, but could not find an explanation that allowed for each digit to represent some of the sticks, because together they would not represent the entire group.

Discussion

The results of the second interview seem to indicate that several of the students were in a "period of transition" regarding their understanding of two-digit number symbols. Clearly their understanding was still immature with regards to digits representing collections of ten ones, but they were able to express more accurate explanations of the value of tens digits with some assistance.

The results of this study are showing that (a) the use of place-value blocks needs careful planning, and (b) students' on-going constructions of meaning for multidigit numbers need to be closely monitored. A teacher of young children cannot assume that students are making sense of number representations in the same way that the teacher is.

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Teaching mathematically gifted students in primary school

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The paper examines the changing conception of giftedness over the last four decades, and characteristics of mathematically gifted students. Ways of identifying mathematically gifted students in a study conducted with year 6 students are described. The Renzulli Enrichment Triad model for catering for gifted students is outlined. Case studies of a number of school and individual teacher's programs which cater for mathematically gifted students are described. A professional development module [CD ROM] designed for teacher inservice will be demonstrated.

Some schools and teachers have spent large amounts of time and energy helping students with learning difficulties and have paid scant attention to those students who demonstrate special talents in a whole range of areas. The exception to this is those students who show exceptional ability or potential or the sporting field. It is reasonable to conclude that we have treated these talented students inequitably when social justice and equity are driving forces in our education systems.

This paper looks at the construct of giftedness and how individual classroom teachers and schools identify gifted students and how they deal with these students in ways which maximise their gifts. It further focuses on gifted mathematics students and ways of identifying them. Classroom programs and whole school programs are discussed with reference, among other things, to the use of technology including the World Wide Web and email.

Definitions of Giftedness

Braggett (1994) indicated that during this century, the notion of giftedness has moved from a unitary concept based on IQ measures to a multidimensional concept (Guilford, 1967) which included a range of abilities and incorporated both products and processes. Renzulli (1977) included Above-average Ability, Creativity and Task Commitment in his analysis of the components of giftedness, thereby indicating the importance of motivation as a determinant.

According to Gagne (1995), giftedness relates to five domains or ability areas - Intellectual, Creative, Socioemotional, Sensori-motor and Other. In his model, giftedness refers to ability while talent refers to performance or behaviour in specific areas. This distinction allows educators to shift the emphasis from 'the gifted' or 'the talented' or from 'being gifted or talented' to the development of gifted behaviours in students showing high potential for benefiting from special educational services (Renzulli & Reis, 1997).

Gardner's (1993) theory of multiple intelligences included linguistic intelligence, musical intelligence, logical mathematical intelligence, spatial intelligence, bodily kinaesthetic, intrapersonal and interpersonal intelligence. This lends support for the multidimensional nature of the concept. After analysing the various theoretical positions, Feldhusen (1989) concluded that giftedness consists of: superior general abilities; special focused aptitudes and/or talents; the acceptance that high level creative production is personally achievable; and the motivation to learn and achieve.

Characteristics of Mathematically Gifted Students

Mathematics educators differ on what constitutes mathematical giftedness. Trafton (1981) claimed that the type of students who are labelled "gifted" and "talented" is not always clear, since the term is loosely applied to three classes of students: (a) those who learn standard content well and perform accurately, but who have difficulty when taught at a faster pace or deeper conceptual level; (b) those who learn more content and at a deeper level, who reason well, and who are capable of solving more complex problems than average students; and (c) those who are highly talented or precocious, performing at the level of students several years older with little or no formal instruction. He seems to be suggesting that he would not consider the first two groups as mathematically gifted.

Greenes (1981) identified seven attributes that characterise mathematically gifted students, namely:(a) spontaneous formulation of problems; (b) flexibility in handling data; (c) data organisation ability; (d) mental agility or fluency of ideas; (e) originality of interpretation; (f) ability to transfer ideas; and (g) ability to generalise. These attributes could be applied to the last two groups mentioned by Trafton (1981). Johnson (1983) suggested that the characteristic that separates a gifted from a non gifted child in mathematics is the quality of the child's thinking; that is how the child reasons mathematically.

The most comprehensive list of characteristics of mathematically gifted students was provided by Lester and Schroeder (1983) in their summary of the work of Russian psychologist, V. A. Krutetskii which follows:

1. Perceive mathematical material in a problem both analytically and synthetically.
2. Generalise quickly and broadly both the content of a problem and the method of solution.
3. Curtail processes for solving similar problems with relatively little exposure.
4. Switch easily from one cognitive process to another even if it is qualitatively different.
5. Are independent of conventional solution techniques and rather readily reconstruct thought patterns.
6. Strive for elegant solutions (i.e., simple, economical solutions which are direct, general, and clear.
7. Reverse reasoning processes easily.
8. Thoroughly investigate aspects of difficult problems before directly attempting to solve them.
9. Remember generalised and curtailed structures associated with problems and their solutions. (This may help explain the sudden flashes of insight often experienced by these students.)
10. Tire both mentally and physically less readily during mathematical activity than during other kinds of lessons.
11. Develop abilities in stages. (Ability to generalise, appears first, curtailed reasoning, interest in elegance, and generalised, curtailed memory later.)
12. View the world mathematically; have a mathematical cast of mind.

It is evident from this discussion that mathematical giftedness has many dimensions which will be exhibited in varying degrees by students in our classrooms. Teachers are faced with the task of identifying those who are potentially gifted so that they can offer them learning experiences that will develop their gifts to the full. In the next section ways of making such identification are examined.

Procedures for the Identification of Mathematically Gifted Students

Since giftedness is widely regarded as a multifaceted construct, Landvogt (1991) recommends choosing a number of identification measures when seeking to identify mathematically gifted students. She suggests including mathematical problem-solving performance, teacher nomination, peer or self nomination, and parent nomination to name a few.

Adibnia (1996), in a study which sought to enhance the problem-solving performance of year 6 students, used two measures of problem-solving ability to identify the 'high ability' or 'gifted' students. The first was a test he devised for measuring problem-solving performance and for which he used an holistic scoring scheme, and the second was the Collis, Romberg and Doig (1992) Mathematical Problem Solving Profiles. Adibnia (1996) also had each classroom teacher complete a teacher-nomination checklist with rated students on a number of criteria related to mathematical giftedness and the pupils were required to fill in a self-rating checklist on their perceptions of themselves and mathematical giftedness compared with their peers. He then was able to rank order the students on all four measures and combine these rankings to gain an overall ranking of the students. From this he selected those in the top 25% to the 'high ability' or 'mathematically gifted' category, those in the bottom 25% in the 'low ability' category and the middle 50% in to the average category.

The Enrichment Triad Model

Renzulli and Reis (1997) have developed a School-wide Enrichment Model [SEM] for identifying and developing those students who exhibit gifted behaviours in many different areas. Their model is based on Renzulli's (1977) Enrichment Triad Model (Figure 1) which is sometimes linked with the Revolving Door Identification Model (Renzulli, Reis, & Smith, 1981) which allows for students to move into and out of Type III activities depending on their level of interest and the like. For Renzulli (1977), enrichment means "experiences or activities that are above and beyond the so called "regular curriculum" (p. 19).

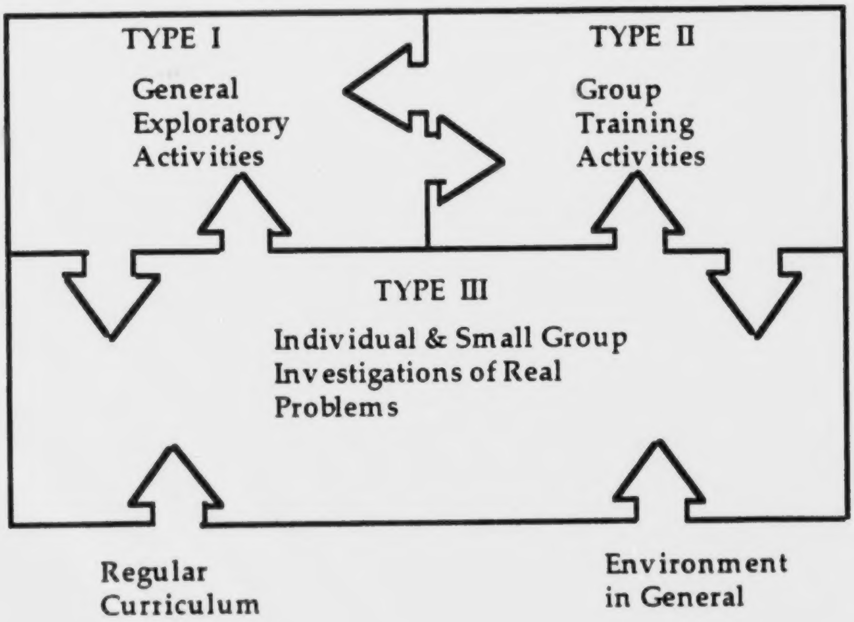
TYPE I
General Exploratory Activities
TYPE II
Group Training Activities
TYPE III
Individual & Small Group Investigations of Real Problems
Regular Curriculum
Environment in General

Figure 1. Enrichment Triad Model (Renzulli, 1977, p. 20)

Type I Enrichment Activities

These are generally available to and appropriate for all students, however, the major objectives for these activities are (i) to assist teachers and students in identifying those topics that some students may follow in depth, and (ii) to assist teachers in making decisions about the Type II enrichment activities that are required for particular students to progress further. Some examples of these Type I activities are:

1. Interest centres where student interest in a topic can be aroused and the student can explore the topic further outside the centre.
2. Listening to and discussing with resource people who are actively and creatively involved in their own professional or artistic field and who could be invited to discuss with potential gifted students their particular area of expertise or interest.
3. Visits to workplaces which involve more than just information gathering although for some students this might be the beginning of their interest. If high ability students can

ask questions and interact with people in their workplace, they can delve more deeply into a particular interest.

4. Students' interests sometimes are awakened through magazines on a particular topic or a video or TV program which covers the topic. With the CD versions of encyclopedia so readily available, some students' interests may be aroused as they explore topics on the computer.

5. There are many specific books on mathematics topics for children from primary level through to tertiary. History of mathematics and famous mathematicians are both interest areas for some students. Sometimes these are found in bookstores or libraries. Some of the journals for mathematics teachers have articles which may stimulate some students.

6. Investigations with mathematics equipment such a pantograph, a planimeter, or a clinometer may lead interested students to dig into the history of measurement and measurement devices.

7. Calculator explorations and computer software of varying types also have the potential to generate interest.

Type II Enrichment- Group Training Activities

These are designed to develop learners' abilities to deal more effectively with content which formed part of the Type I activities. Most Type II Enrichment involves training activities or process-oriented teaching activities which major on the development of thinking and feeling processes. Their focus is on developing specific skills involved in: (a) creative thinking, (b) critical thinking, (c) reflective thinking, (d) independent study, (e) divergent thinking, (f) inquiry training, (g) self-directed learning, (h) sensitivity training, (i) lateral thinking, (j) research, (k) inductive reasoning, (l) problem solving, and (m) awareness development. In fact, the higher levels of the Bloom, Hastings, and Madaus taxonomy of educational objectives (1971), namely, analysis, synthesis, and evaluation are often considered when looking at Type II Enrichment Activities.

While the focus in these types of enrichment activities are on developing processes, it is important to remember that this process development is a means to an end, rather than an end in itself. The experiences should arise out of the genuine interests that students exhibit in the Type I experiences and also prepare them for the Type III Enrichment which flows on from Type II activities.

Type III Enrichment - Individual and Small Group Investigations

These consist of activities in which learners' investigate real problems or topics. They use the skills developed in Type II activities to formulate a problem to be investigated and the methods to apply in dealing with its solution. To be an authentic Type III activity the students should be involved in investigation/research/production similar to a professional working in the same field. The outcome of such an activity should be the generating of a real product to be reported to a real audience.

Renzulli (1977) claims that the teacher's role in Type III Enrichment is more managerial as a resource person whose job is to help identify a student's interest and then to encourage the student to pursue this interest in an investigative way with a view to a creative/productive outcome. Once the student has refined the focus of his/her interest the teacher can then assist in the selection of a real audience to which the product of the investigation can be presented in whatever form.

Examples of School Programs Catering for Gifted Students

Secondary School

Two schools which have taken steps to cater for students with high ability some of whom may be classed as mathematically gifted have done so in different ways to allow for the different programs operating in their school. One is a country high school which saw the need to cater for a large range of ability in year 8 mathematics students. In order to accomplish this, it was decided to arrange clusters of two or three classes to be timetabled concurrently. Year 7 data on the students' performance on problem solving, computational skills, general interest areas and possible behaviour management problems were used to allocate students to the various class groupings. In this way, both the high ability and the low ability and special needs students received learning experiences which were appropriate to their needs. A group of teachers who opted to take part in this program worked to cover the regular curriculum content but also developed appropriate resources which provided extension activities for the high ability students. Evaluation by the students, teachers and parents have been quite positive and there were plans to extend the scheme to other year 8 subjects and into years 9 and 10.

Primary School

The second school is a parochial school in a large city in tropical Queensland which has adopted a policy of flexible purposeful grouping as well as home classes across a number of grades. This is accompanied by workshops where gifted students are withdrawn for a whole day to cover specific activities in areas such as drama and writing. Students are placed into four mathematics levels at the beginning of the year based on testing of the previous years content as well as some off-level testing. There are two high ability groups, one or middle ability groups, and a lower ability group which is kept small [12-15] to allow the students who need the extra attention to receive it. Students' performance, effort and ability are constantly monitored and students can be moved to a higher or lower level in the light of these data. Students who feel they are not handling a particular level can also request to move to a lower level when this occurs. This is an important principle that the school is strongly committed to and it means that students do not feel that they are trapped in the same level for a whole year. Assessment is continuous and, because there are home class mathematics lessons as well as lessons in the flexible grouped arrangement, it is shared between the regular class teacher and the teacher in the particular ability level.

The focus of mathematics learning in the high ability group is on mathematical processes and thinking skills. Problem solving with universal applications and divergent thinking are developed through individual and small group investigations with materials from the Maths Task Centre Kit (Curriculum Corporation, 1993), the Mathematics Curriculum and Teaching Program [MCTP] (Lovitt & Clarke, 1988) and from materials used to prepare students for the Mathematics Olympiad and the various mathematics competitions such as Westpac and for activities relating to the annual Tournament of Minds competition. However, when high ability students are in their regular classes the teachers use curriculum compacting as a way to buy time for these students to use on extension activities.

Examples of Individual Classroom Programs for Gifted Students

Lower Primary School:

One example of how a teacher caters for gifted students within her classroom is taken from another school in the city mentioned earlier in tropical Queensland. The teacher has a year 1, 2, and 3 multi-age class which includes children ranging from gifted to intellectually impaired. Her formal teaching focuses on a particular concept with planning of activities to develop the concept based on the various levels of Bloom et. al. (1971) taxonomy of educational objectives and Gardner's (1993) theory of multiple intelligences. In this way she caters for the different ability levels within this class grouping.

Two other strategies used to individualise instruction are learning centres with a full range of activities based again on Bloom's taxonomy and projects whose topics are chosen voluntarily by the children. A problem-solving approach is the most common strategy for project work. Students who tested out of the current mathematics unit chose to identify the most fierce dinosaur and then identified height, mass, teeth, claws, and speed as key factors which they would need to research. As a conclusion to their project they wrote a report and presented their findings to the class.

The school also enters students in the Tournament of Minds competition and, in 1996, allowed younger students to be part of the teams with older students. In 1995 students in this grade won a prize for a big book they wrote as an entry in the Queensland Association of Mathematics Teachers competition.

Upper Primary School:

Joshua was in year 6 in 1996 at a small urban school which was located in a low socio-economic area of the same tropical city mentioned above. He was found to be gifted in literacy, numeracy and computer technology and in his second year at school learned Japanese with year 6 and 7 students and in his year 6 class was learning Japanese at a level way above his grade level. He was the resident expert on computers and was called upon to assist teachers with their software and hardware difficulties. He used multimedia tools to produce his reports which include audio and video segments which he developed and programmed.

In mathematics Joshua spent a great deal of his time pursuing individual projects and investigations like those in the Maths Task Centre Kit (Curriculum Corporation, 1993), but he was also always involved in group investigations in mathematics within his own class where he was a popular leader.

Resources for Assisting Teachers of Mathematically Gifted Students

As mentioned earlier, there are a number of commercially available kits such as the MCTP (Lovitt & Clarke, 1988) materials and the Maths Task Centre Kit which are excellent for extending mathematically gifted students. These can be used with all students at some point in their mathematics learning but also serve the purpose well as extension and enrichment for gifted students.

Mathematics competitions such as the Westpac Maths competition, the Queensland Association of Mathematics Teachers competition and the Mathematics Olympiad can serve as a valuable avenue for catering for gifted students. Locally run mathematics camps can often be an opportunity for Type I activities where students' interests in environmental or outdoor mathematics may begin.

Another medium of growing importance in catering for gifted students is the Internet. There are many locations on the WWW devoted to mathematics for school students and these sites are generally excellent sources for teachers who are seeking new materials and ideas for broadening their program for gifted students. A brief listing of some URL's found to be of value for mathematically gifted students follows:

Maths Lessons
http://forum.swarthmore.edu/~steve/steve/mathlessons.html
http://www.eskimo.com/~user/zcurric.html
http://www.mcrel.org/connect/lesson.html#math
http://carson.enc.org:80/lesmath.htm
Maths Association of Victoria HOME PAGE
http://www.srl.rmit.edu.au/mav/
QLD ASSN of Gifted & Talented
http://www.maxlink.com.au/qagtc/welcome.htm
Internet Hot Spots
http://www.srl.rmit.edu.au/mav/intsites/index.html
OZ- Gifted
http://owl.qut.edu.au/common-cgi-bin/forum/OZ-GIFTED/
Fibonacci Numbers and Nature
http://www.ee.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html
Activities for classroom on pi
http://www.ncsa.uiuc.edu:80/edu/RSE/RSEorange/Piactivities.html
Problems set out in grade levels solutions posted each week
http://www.hmco.com/school/math/brain/
http://www.pacificnet.net/~mandel/Math.html
Classroom, the Twelve Days of Christmas
http://www.feist.com/~lshiney/lesson/math.html
Great photograph activities all maths related
http://acorn.educ.nottingham.ac.uk/Maths/photomath/

Union Elementary School - collection of problems each week
http://plainfield.bypass.com/~union/problems.html
Maths competitions and clubs
http://cq-pan.cqu.edu.au/schools/smad/mathcomp.html
Wonderful World of Mathematics
http://plainsfield.bypass.com/~union/world.html
Mega Mathematics
http://www.c3.lanl.gov/mega-math/menu.html
Houghton Mifflin Education Place main index.
http://www.hmco.com/hmco/school/main.html
Uni of Connecticut
http://www.ucc.uconn.edu:80/~wwwgt/
Renzulli Centre
http://www.ucc.uconn.edu:80/~wwwgt/nrcgt.html
Wyoming Future Problem Solving Program
http://www.uwyo.edu/a&s/nasc/nasc2.htm

Some teachers have taken advantage of the WWW to introduce their potentially gifted students to Type I activities and then the students have been able to carry out their own investigations using the WWW. It seems that the opportunities are almost unlimited for us as teachers if we can explore this resource for and with our students.

Conclusion

In this paper an attempt has been made to cover a number of areas related to teaching mathematically gifted students. One of the main challenges for classroom teachers is to find time to collect resources which will extend their gifted students to their fullest potential. The author has produced a professional development package on *Teaching Mathematically Gifted Students* which is delivered on CD ROM to people in their own home or workplace.

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Orchestrating Different Voices in Student Talk about Infinity: Theoretical and Empirical Analyses.

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In this paper we review major theoretical perspectives on differences between students in their thinking and classroom talk, and conclude that Bakhtin's concept of *voice* provides a powerful way of theorising difference that draws attention to the multivocality (inherent diversity) of individual ideas and utterances. It is the orchestration of the diversity of voices that enables new insights to emerge both collectively and individually. We apply analytical tools (derived from Bakhtin) to an episode of collective argumentation focussed on the concept of infinity, and we identify key elements of everyday classroom practice that enable challenging and productive talk to occur.

There is considerable research evidence for the view that specific types of student talk - talk that involves explaining, justifying and representing ideas for others - enables the appropriation of mathematical ways of thinking. Such talk requires certain social conditions to be established within classrooms - social conditions that support students to engage in joint activities, that support students to propose and consider different viewpoints, that support students to explain and justify their ideas to each other, and that support students to evaluate the relative worth of competing proposals. Recently we reviewed evidence arising from our own research programme which indicated that such classroom conditions can be facilitated by following the ground rules and social values inherent in collective argumentation (see Brown & Renshaw,1995;1996; Renshaw & Brown, 1997). Our purpose here is to begin to examine in more detail, using Bakhtin's notion of voice, the dialogues that occur between students during episodes of collective argumentation. In particular, we focus on differences that arise between students as they try to communicate and represent (in words and diagrams) their ideas to each other. We present a range of evidence for the claim, that it is the orchestration of differences in students' talk about mathematics tasks that is generative of new insights and deeper understanding. Orchestration is a key term in our paper because it conveys our view that understanding is a process that involves learning to speak in different voices and being able to move flexibly and smoothly between different voices in order to communicate ideas and convince others of their relevance. Orchestration also conveys the notion that understanding is not achieved by creating insight in isolation, but is achieved by composing ideas with others, learning to speak with others, and learning to incorporate aspects of their performance in ones own utterances.

Theoretical Framing of Differences

Social conflict and cognitive disequilibrium. The view that differences between students are crucial in creating the conditions for change is supported by Piagetian and neo-Piagetian literature which has emphasised the role of social conflict in creating cognitive disequilibrium and conceptual change. Relevant research reviewed by Chapman and McBride (1992) has suggested, however, that conflict per se does not relate to conceptual change. Rather, participation in argumentation - discourse that has the form of a dialogue in which claims are made and challenged by appeal to evidence and reasoning - appears to be the social process that generates cognitive change. This conclusion is supported by research from the cooperative learning literature. Johnson and Johnson (1987) investigated two contrasting formats for group interaction -*concurrence* and *controversy*. The process of concurrence involves maintaining social cohesion within the group by quickly resolving differences and disagreements, whereas the process of controversy involves the exploration and analysis of differences prior to moving towards a joint solution. Research has shown that the process of controversy is productive of deeper understanding than the process of concurrence.

The work of Marton, Asplund-Carlsson and Halasz (1992) is relevant here - though not framed by a cognitive-developmental theory. They focus on intra-individual processes that appear to enable individuals to make progress in problem-solving. The first process - reflective variation - involves considering and alternating between different interpretations of a task. The second process - elaborative variation - describes how particular interpretations are followed and implications are explicated and compared. Their work is consistent with the viewpoint being articulated in this paper, namely, that it is the orchestration of differences in the social domain that enables progress in thinking to occur. Later, we refer to research by Wistedt and Martinsson (1996) who have employed Marton et al's distinctions in considering the different *voices* that can be heard during student discussions.

A sociocultural view - exploratory talk. Argumentation, involving attempts to publicly justify opinions and ideas, needs to be distinguished from quarrelling on the one hand, where the participants simply disagree, and from conflict resolution on the other, where the participants' overriding goal is to reach consensus (concurrence) rather than to search for the best solution to the problem (Chapman & McBride, 1992). Similar distinctions have been proposed also in the neo-Vygotskian literature by Mercer (1995) who defined three social modes of thinking. The first social mode of thinking, *disputational talk*, is characterised by disagreement, assertion and counter-assertion by students as they attempt to work together on a task. Differences in viewpoint, however, do not lead to a pooling of resources or constructive criticism and analysis, but rather to the entrenchment of students within their individual perspectives. At the other pole in Mercer's typology is *cumulative talk*, which is characterised by uncritical compiling and accumulation of ideas. Such talk is equivalent to concurrence-seeking as defined by Johnson and Johnson, where students confirm and repeat each others' contributions as a means of maintaining harmony. So while conflicts are avoided or quickly resolved, there is no attempt to analyse or evaluate the worth of the contributions.

The third social mode of thinking is *exploratory talk*, which is similar to the notion of argumentation and the process of controversy. Talk in this mode is characterised by constructive and critical engagement with the ideas proposed by the partners in the group. Ideas are challenged and defended by proposing justifications, explanations and alternatives. In exploratory talk, claims are publicly accountable and reasoning is visible and therefore open to evaluation and critique. Mercer's neo-Vygotskian theoretical framework emphasises the need for students to gradually appropriate the ground rules of exploratory talk so that their own thinking takes on a dialogical and self-argumentative form. So, engagement in exploratory talk or argumentative discourse, is theorised to have quite general cognitive effects by both neo-Piagetians and neo-Vygotskians, the former describing the effects in terms of cognitive restructuring, and the latter in terms of the internalisation of social forms and processes.

A sociocultural view - collective argumentation. In previous papers (Brown,1994; Brown &Renshaw,1995; 1996; Renshaw & Brown,1997) we have reported research on a process akin to exploratory talk, namely, *collective argumentation*. In one study (Brown, 1994) we assessed the talk of students in collective argumentation using the analytical device of *transacts*, which gives a measure of the extent to which students' talk, and specifically their explanations and justifications, makes reference to previous contributions in the discussion. Using transacts as a indicator of the quality of argumentation within groups, we found that students showing most change in their mathematical thinking were in groups where talk was of a higher quality and more responsive to the partners' ideas.

While our purpose is not to conduct a reductive analysis of the sub-components of collective argumentation, it is clear that the first phase of collective argumentation is important for differences between students' ideas to be revealed and explored in talk. Collective argumentation involves students working in small groups (varying from 2 to 5 members per group) where initially they individually "represent" a problem (using pictures, diagrams, drawings, graphs, algorithms, numbers etc) and then "compare" their representations with those of other group members. This phase of individual representation and comparison provides the potential for differences in understanding to be exposed and examined. Subsequent talk by the students regarding the appraisal of representations is guided by the keywords - "explain", "justify", "agree". Finally, moving from the small group to the classroom group, the thinking within each group is validated as it is presented to the whole class for discussion. Evidence from disparate theoretical traditions indicate, therefore, that differences in thinking between students creates the context for productive talk, but the theoretical terms that should be used to describe such differences, and what types and levels of difference optimise learning remain issues of contention.

Differences That Facilitate Understanding

Cognitive-developmental tradition. Piagetian and neo-Piagetian research has been guided by a stage developmental theory in describing differences between students. With regard to their understanding of a specific concept, like number or weight, students have been classified as non-conservers, transitionals, or conservers. Alternatively, across a range of concepts ("decalage"), students can be developmentally classified as transitional between pre-operational and concrete-operational thinking if they show conservation of only a few concepts, or as concrete-operational thinkers if they conserve a range of concepts across a range of materials (continuous and discontinuous materials, regular and irregular objects). Finally, they might be classified as formal thinkers if they show more generalised, hypothetical and abstract thinking. The units of analysis here are the students (classified according to stage) rather than ideas, the assumption being that students are restricted in their thinking according to hierarchically organised stages. Change for students has been found to be optimal when the partners in the discussion are "one-stage different".

Interweaving concrete and abstract concepts. Our own research on collective argumentation has been framed by sociocultural theory which opens up a number of alternative ways to describe differences between students, and to consider when differences are optimal. Initially, we drew on Vygotsky's work which emphasised the contextual embeddedness of conceptual understanding. In everyday activities at school and home, children use concepts with the assistance of partners, who can enable them to apply concepts in quite abstract and general ways prior to gaining conscious awareness and personal control. Thus, at a particular point in time, children may engage in various forms of conceptual thinking depending on the challenges and support offered by others. Here, difference in conceptual thinking is not determined by internal structures, but is dynamic and created interactively as students work with ideas introduced by the participants in the dialogue.

During collective argumentation, we've found that students will represent a problem in quite disparate ways. Some students, in an attempt to represent a problem mathematically, draw a diagram that foregrounds irrelevant and marginal features. A partner might draw a similar concrete representation but s/he appears to have a better grasp of the mathematical significance of the drawing. A few students will work with quite abstract and sophisticated representations that summarise and foreground the central mathematical details of the problem. To coordinate their actions as they work towards a joint representation, students must speak and act in ways that are responsive to the words and actions of the other participants. In such dialogues, individual contributions can be paraphrased, reinterpreted, or particular words replaced by more general terms. In this manner, collaborative interaction can enable students to enter into more abstract and general ways of speaking about, and acting towards objects - that is, it can open up a zone of proximal development. Deeper understanding does not require replacement of the more local and concrete concepts with the more abstract and general concepts, but the weaving together of these two forms in a process of mutual transformation. In summary, the Vygotskian perspective frames difference as a contrast between the local personal experiential and concrete versus the general abstract and formal. However, progress in thinking requires the integration of these two forms of knowing rather than the replacement of one form by the other.

Voice - speaking as a member of diverse language communities. Recently we have begun to move beyond Vygotsky's focus on weaving together everyday and scientific concepts, to explore Bakhtin's theory of *voice*. He formulated a theory of language which emphasised the active, situated, and functional nature of speech as it is employed by various communities within a particular society. In conversation with others, *personal voice* is given to words that are employed by a wider community of users. So, in any individual utterance, a number of *voices* can be heard - the voice of the particular person but also traces of the voices of other community members who had previously employed those words to convey their own meanings. This provides a powerful method of theorising difference. Difference, diversity, or multivocality occurs not only between speech partners, but within individual utterances as speakers use the words and phrases of others to construct and convey an idea. Who is the author of the idea? At one level it is the speaker, but the speaker depends on the appropriation of others' contributions to not only compose the idea itself but to convey it to others. So, insight and understanding arises from the orchestration of difference - the effective composing and recomposing of different voices that produces a performance (explanation, justification, solution to a problem, etc) that is appreciated and accepted as valid by the community.

Privileged voices - everyday, school, and mathematics voices. Schooling is not neutral, however, in providing spaces for all student voices. Schooling privileges particular ways of talking. For example, Wistedt and Martinsson (1996) analysed a session of joint problem solving by groups of 11 year-olds who were working on the problem of dividing 100 into three equal parts and expressing their answer as a decimal. Different voices were identified in the children's discussions - including the discordant everyday voice of practical reasoning - *"But why do we have to make three bookstands. He just needs two, one on each side of the books."* This student attempts to speak outside the privileged voice of schooling - redefining the problem so that the dilemma is resolved practically rather than mathematically. His utterance might also be read as a critique of the task students are asked to solve in schools. The student's partner reasserts the voice of schooling by saying - *"Yes, but now they've made it in this weird way so we will have to solve it as it is, won't we."* This student argues that they have to accept the premises of school mathematics tasks, that is, the purpose of the task is not primarily practical (producing actual objects such a bookstands), but educational - to learn strategies and concepts related to division and decimals. Note also that she makes clear her solidarity with the first speaker by using the contrast of *"them"* versus *"us"* (*"they've made it in this weird way so we)* , and distancing herself somewhat from the schooling voice by noting it was *"weird"*. The subtle power of this utterance is to both reject the first speaker's practical voice and to invite his participation in her proposed new voice of school mathematics. In this way, the practical voice was marginalised and the students turned their attention back to the task - now interpreted as a school task where the major concern was to produce an acceptable answer. This school voice can be heard in the comment by one student *"I don't think you can solve this task if you're not allowed to write 33 plus 33 plus 34"* Here the student's major concern is to produce an acceptable answer - *"if you're not allowed. to write..."* The term *"allowed "* has connotations of following directions and relying on an authority (the teacher) to determine correct from incorrect ideas - in other words the traditional voice of schooling.

The move from a school mathematics voice per se to a mathematics voice per se is quite subtle, but it occurs as the students move beyond their concern to simply produce an acceptable answer, and begin exploring and speculating about the nature and limitations of the number system. Wistedt and Martinsson provide examples of the students employing the voice of mathematics as they face the puzzling problem of expressing one-third as a decimal - 0.33. They realise that the number system has limitations in expressing certain quantities - *"You can go on forever. It will never come out even. But the more 03's you write, the closer you will get"* . This also provides the space for them to reconstruct the notion of error or remainder - *"33.33 plus 33.33 plus 33.33 and then you add up ... you'll get a nine here and another nine and another... and then plus 0.01 And then it will come out even. There will be a small error that we will have to add"*. Note the student's use of the pronoun "*you*" which indicates that she is speaking in the general voice of the mathematician - that is the procedures and outcomes she is reporting are not personal opinions but general claims about the nature of such problems.

Analysis of a Fragment of Collective Argumentation

The notion of *multiple voices* seems particularly powerful in capturing the flow of dialogue between students, and providing an insight into the way differences in the dialogue are integrated - orchestrated - to enable new understandings to be co-constructed. To examine these issues further we turn to an analysis of an episode of collective argumentation where the problem of representing infinity is addressed by students in their groups. The students in Wistedt and Martinsson's study had visited the notion of infinity when they discovered the recurring decimal in expressing one-third. In our study, the notion of infinity was introduced explicitly so we could observe the way that children co-constructed arguments to support their particular representation of infinity, and to identify the different voices that could be heard during their discussions. In this paper, we focus particularly on the last phase of collective argumentation where students are recounting and presenting their thinking to the whole class for discussion and validation.

The students were in Year 7 from a Brisbane primary school, and had used collective argumentation as part of their classroom learning processes across all curriculum areas for more than two school terms. In this particular episode they were asked to co-construct a definition of infinity that they could share with the class. The majority of the interaction below occurred after 15 minutes of individual representation and small group discussion, when the teacher invited groups to share their responses with the class.

The Transcript and Analysis

Section 1: Turns 1 - 6

Prior talk involved the teacher introducing the topic, with a number of question-answer sequences between teacher and students. Groups then worked on the problem of representing infinity for about 15 minutes prior to class presentations

Susan, Andrea and Kelly come out to present to the class. They have represented 'infinity' by drawing a line segment on the whiteboard.

1 Susan: In the dictionary infinity means an infinite number, or time . . . and we thought the soul was infinite, it would go . . . If this line was infinite it would go forever, and God's love for his people is infinite because he will love us forever.
2 Teacher: What do you think of that class?
3 Anne: The line is wrong because. . . the line (that the group has drawn) has two ends.
4 Simon: It's not a true line . . . a line goes on forever.
5 Teacher: How could we represent that? I understand your point, saying that this (referring to the group's line segment) is not a line, it's a line segment. How could we represent an actual line?
6 Anne: You could draw a circle.

In this section of transcript (Turns 1-6) the students and teacher introduce a number of privileged voices, that actually open up alternative possibilities for the direction of the dialogue. There is the authoritative and abstract voice of the dictionary, which enters via Sarah's reading of the definition. Talk of the "soul" and "god" makes the voice of religion audible (this arises because "infinity" is studied in this Catholic primary school both in religious lessons and in mathematics), while the drawing of the line segment and the comment "if this line were infinite it would go on forever" brings the voice of mathematics into the conversation. Anne's comment (*"The line is wrong"*) directs the attention of the students primarily to the mathematics of the situation, and away from exploring further the other possibilities. At Turn 6, Anne's suggestion that a circle would be a good representation of infinity reveals why she rejected the line - a line has two ends while a circle has no ends.

Section 2: Turns 7 - 17

Another group, Carey, Melanie and Margaret approached the black-board and drew a circle, and explained that it goes on and on like time. Ten turns at talk involving the teacher and other children clarified the idea that a circle or any closed figure could represent infinity because the line never ends and can be overlapped an infinite number of times. Then the teacher summarised their talk and recalled the line segment introduced by Susan

7 Teacher: Going back to Susan's idea. How could we take that line segment and represent it as an infinite line? With no beginning and no end. One way is with a circle, but as Simon said, there seems to be something missing. Susan, do you want to take back control of the conversation.
8 Susan: Bonnie.
9 Bonnie: Well I did a spiral and it keeps on going around and around and when it finishes, it doesn't actually finish. It keeps on going around when it finishes, it just goes around again.
10 Susan: Margaret.
11 Margaret: You could put two arrows on it (the line segment) to show that it goes on forever.
12 Teacher: Would you like to do that? (To Susan)
Susan and Kelly put an arrow on each end of the line segment.
13 Teacher: What's that represent Margaret?
14 Margaret: It just represents that a line can keep on going that way (points) and that way (points), so that it doesn't end.
15 Teacher: It doesn't have a . . .?
16 Margaret: It doesn't have a beginning and it doesn't have an end.
17 Teacher: So, what are some of the important qualities of the idea of infinity? (To class)

In this section of the transcript (Turns 7 to 17) the students are guided to consider the various representations of the concept of infinity proposed so far - circles, closed figures of any shape, spirals, and lines. In clarifying how a line segment might be drawn to better convey the idea of infinity, Margaret suggests adding two arrows at each end to indicate that the line goes on forever. This enables Margaret, with some support from the teacher, to highlight that infinity has neither a beginning nor an end. Discussion and appraisal of the different representations seems to sharpen the students' awareness of the details of the concept - not only doesn't it end, but it also doesn't have a beginning.

Section 3 Turns 18 - 22

At the end of this discussion and summary Simon and Angela came out to present their ideas.

18 Angela: We drew a clock and we had, um, about, an infinite number of handles, because time goes on for an infinity. That's how we represented that, because time goes on.
19 Teacher: I didn't understand that phrase, could you say it again please.

20 Angela: Well, we drew a clock and we had an infinite amount of handles, the little things that go around, because time never stops. It just keeps going around.

Teacher clarified with Simon and Angela the term for the hands of the clock, and then Angela continued

21 Angela: Time has no beginning and no end like numbers. And we had the dictionary meaning which says this - infinity has the state of being infinite, infinity of the universe, infinity of space, time, quantity - so infinite space, so, it's so that you can't describe it. Um, (infinite) mass is the concept of increasing (mass) without volume, so we thought that we would make a meaning of our own. So we thought that infinity means everlasting number, object and the universe. So infinity is an everlasting thing.

Teacher recalled the key ideas from Angela's presentation.

22 Angela: Infinity can(not) be determined or explained over a vast amount or period of time, because it is an everlasting idea. And I made this up. I think the word infinity is similar to life. No one can fully explain it and just like infinity it has many definitions. We can't really explain life and we can't really explain the word infinity

The contribution of Angela and Simon (Turns 18 to 22) to the discussion of infinity reveals the capacity of students in this class to work with multiple representations and to move between them flexibly - privileging one voice or another depending on their purpose. Angela and Simon reported four representations including a drawing of a clock with a large number of hands, a dictionary definition, a personal paraphrase of the dictionary, and in Turn 22, Angela adds an analogy she has constructed - infinity is like life - because neither concept can be adequately explained. In this episode, we can hear in the student voices that of the epistemologist . Angela not only could entertain multiple representations of infinity, but had begun to consider the notion that any representation of an abstract idea such as infinity is inadequate. This voice is heard first in her talk about the dictionary definition "*...so, it's so that you can't describe it*" This indicates that the dictionary definition was not much help in building an understanding. She reports that they then translated the dictionary terms into their own words, "*...so we thought that we would make a meaning of our own*" They substituted the term 'everlasting' for 'infinity' which is a small change but one that in Jay Lemke's terms is crucial in "making texts talk" and linking abstract and minimalist definitions to personal understanding. However, for Angela the term *'everlasting'* retains similar connotations of mystery and inexplicability as the term *'infinity'* - to paraphrase Angela in Turn 22 - "infinity like life is an everlasting idea that can't be explained".

The voice of the epistemologist is heard in Angela's recount of her group's work as she reports the multiple representations that she and Simon had composed and compared in terms of adequacy and accessibility. In addition to this overarching voice, Angela articulates a series of voices moving from the transituational voice of the dictionary, to the social voice of her group - *"so we thought that we would make a meaning of our own"* - then to the personal voice - "*..And I made this up. I think the word infinity is similar to life.*" However, she doesn't stay within the personal voice. After claiming her authorship of the idea, she moves into the generalised voice of the expert (scientist, philosopher, mathematician) - "*No one can fully explain it and just like infinity it has many definitions. We can't really explain life and we can't really explain the word infinity*". "No one" and "We" convey Angela's intention to speak authoritatively not on behalf of her small group, or personally, but generally on behalf of humankind.

Conclusion

The effort to translate ideas into ones own words, to make personal sense of the concept of infinity, did not happen by chance. The collective ways of thinking in the class have been scaffolded by the teacher over a period of months as he introduced and supported the students in using collective argumentation. There is specific evidence of this scaffolding process in Angela's talk where she uses the word "so" which appears to be a ventriloquation of the teacher. The teacher often prefaced his transformation of student contributions (summarising or paraphrasing) with the word "so". For example, he says " *So infinity means to us an everlasting number*" There were five such occurrences in this episode of collective argumentation (not all are included in the fragment of transcript). Angela also uses "so" as a marker of changing voice. While apparently insignificant in itself, the use of "so" by both the teacher and Angela to mark a change in voice, suggests that the students in this class have begun to move beyond reproductive learning where they rely on an expert to validate their answers, to a co-constructivist view of learning in which a diverse voices can be heard as students try collectively to compose and orchestrate their local understanding of mathematical ideas

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Measuring Attitudes towards Mathematics in Early Childhood and Primary Teacher Education

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Changes in attitude have been measured over a semester of mathematics in the first year of Early Childhood and Primary teacher training. The unit of study is not a curriculum unit, is taught by mathematicians and explores mathematical ideas and experiences. The Fennema-Sherman Attitudes Scale was used to measure changes in *confidence, effectance motivation,* and *usefulness*. Analysis of the results indicates a challenging outcome - the only significant change was a drop in their perception of the usefulness of mathematics.

1 Introduction

A unit of study in mathematics, designed for Early Childhood (including Childcare) and Primary trainees, has been evaluated specifically to measure changes in attitude which occur during the semester. The unit of study is unusual in that it has been designed and is lectured by mathematicians from the science faculty, with advice and workshop support provided by the education faculty. The unit was designed for four main purposes: (i) to develop an awareness of a range of mathematical ideas and experiences not usually studied at school. (ii) to have some success in doing mathematics, (iii) to instill confidence in doing mathematics, and (iv) to develop a positive attitude towards mathematics. Teaching mathematics to such a diverse group can be a daunting task.

Points (i) and (ii), above, are readily measured and evaluated using assignment, workshop, and examination results. However, points (iii) and (iv) above reflect attitudinal factors which are just as important to measure if the unit is to be evaluated for effectiveness and subsequent development. The instrument used to do this was the Fennema-Sherman Attitudes Scale (Fennema & Sherman, 1976) and was administered during the first and last week of Semester 2, 1997. This instrument was used to measure changes in attitude to mathematics in relation to the three factors of *confidence*, *effectance motivation*, and *usefulness*. Note that effectance motivation is a measure of attitude to problem solving and ranges from a lack of involvement through to active enjoyment in seeking challenges.

The unit of study, *Mathematics for Teachers,* is a compulsory first year unit in the three year Bachelor of Early Childhood Studies (childcare) and the four year Bachelor of Education (primary and early childhood). It includes a study of topics selected from: mathematical reasoning, logic, numeration systems, elementary number theory, geometry, and topology, with problem solving providing an overall theme. Weekly contact hours for a student involve two hours of lectures and two hours of practical workshop activities which include hands-on experience with materials that reinforce the topics. Assessment of the unit is by (i) written assignment, (ii) participation in the workshop (problem solving tasks, group work, and group discussions), and (iii) final examination.

2 Motivation

Against broad statistics which reveal that mathematics is often viewed negatively by scholars (Mullis, 1992) is the concern about teacher attitudes to mathematics. Their beliefs and conceptions not only play a role in the methods of instruction of teachers (Thompson, 1992), but are linked to the attitudes and achievement of their pupils (Kulm, 1980). Reports of negative attitudes of prospective teachers towards mathematics, for example in Carroll (1994) and Way and Relich (1993), are therefore cause for concern.

Some studies indicate that it is difficult to change those attitudes. Schuck (1996) describes the beliefs and attitudes, which students bring with them into pre-service training, as chains that are not easily broken. Way and Relich (1993) suggest that perhaps attitudinal changes induced in pre-service training may not be enduring.

Nevertheless, studies like those of Bobis & Cusworth (1997), Mayers (1994), and Perry et al (1994) give hope that the nature of tertiary exposure to mathematics can indeed play a role in improving attitudes. This indicates that we should seek opportunities for fostering improvement, and that there is reason to be positive about the role tertiary mathematics studies can play. Furthermore, we must accept responsibility for ensuring that such studies do not reinforce any existing negative attitudes.

With these goals in mind this early quantitative study was undertaken. The motives were therefore:

- to establish a clear profile of the students in the *Mathematics for Teachers* unit at the outset: age, gender, mathematical background, intended level of teaching, and most importantly, their attitudes towards mathematics;
- to measure any change in those attitudes at the end of this unit;
- to establish any significant relationships between those attitudes and other factors;
- to identify directions the unit should take to support improvement in attitudes; and
- to pave the way for follow-up exploration, both quantitative and qualitative, based on interviews and case studies.

3 Methodology

To measure these attitudes a questionnaire was administrated in the first lecture of the semester and another one in the last lecture of the semester. The initial questionnaire was in two parts. The first covered demographic information such as age, gender, degree in which enrolled, high school mathematics level and two other questions of interest: the grade the student expected to receive and if they had had a choice would they study *Mathematics for Teachers*. The age was recorded in intervals of 17-18 (school leavers), 19-23, and 24 and over, representing the mature age students.

The second section contained a measure of the student's attitude to mathematics. Because of the impact of the Fennema-Sherman scales, which have "been felt widely in all research on attitudes toward mathematics" (McLeod, 1994, p639), a selection of these was chosen. *Mathematics for Teachers* is neither a methods of teaching nor a curriculum unit and we specifically wished to measure the changes in attitudes associated with studying this unit. It was decided not to use any of the available scales of Ludlow and Bell (1996), Nisbet (1991) or Relich, Way and Martin (1994) as these scales have been specifically designed to measure attitude to teaching mathematics as well as measuring an attitude to mathematics. The scales of *Confidence in Learning Mathematics*, *Effectance Motivation in Mathematics* and *Usefulness of Mathematics Scale* were chosen for the following reasons.

Prior experiences in mathematics form the strongest "chain" that Schuck (1996) sees as responsible for inhibiting a student's progress in mathematics. Bodis and Cusworth (1997) and Sax (1994) have noted that previous mathematical experience has an impact on a student's confidence in mathematics. McLeod (1994) feels confidence in learning mathematics can also indicate a "belief about oneself" (p641), whereas Meyer & Koehler (1990) and Relich, Way and Martin (1994) see confidence as being a part of a student's self-concept of mathematics. As such the scale measuring *Confidence in Learning Mathematics* was included.

The area of emotions are generally neglected in affective domain research, according to McLeod (1994). The "Aha!" experience (McLeod, 1994) to problem solving is the reaction all educators would like to see in their students, but unfortunately most students do not see mathematics as a subject beyond arithmetic or a set of rules that will quickly give a correct answer. (Carroll, 1994; McLeod, 1994; Schuck, 1996; Southwell & Khamis, 1994). The lack of perseverance, motivation and involvement in problem solving is cause for concern. Therefore the scale *Effectance Motivation in Mathematics* was selected.

Mathematics participation is strongly associated with a perception of usefulness of mathematics (Carroll, 1994; Leder, 1992; Meyer & Koehler, 1990; Schuck, 1996). Armstrong & Price (in Leder, 1992) found that usefulness of mathematics was the most important variable for high school senior females. The occupational plans of students also reflect their perception of usefulness of mathematics (Leder, 1992). Since the pre-service teacher's course has a high proportion of females of which a large number here are enrolled in Early Childhood and Childcare degrees, the effect of usefulness of mathematics should not be ignored. Consequently the *Usefulness of Mathematics* scale was included.

Some wording was changed in the scales because the educational institution is a university, not a school. The follow up questionnaire contained: the grade expected question, the question of choice in studying the unit, and the three attitude scales.

The student's total for each scale on each survey was considered to be a separate variable. The difference between each scale total was to be measured and this difference tested for any significance. Demographic variables were also used as factors to determine if there was any difference between or within the groups of students (Kulm, 1980).

4 Data and Findings

There were 298 students originally enrolled in *Mathematics for Teachers* for the semester. A total of 237 completed the pre-test survey at the beginning of the semester. Only 160 students completed the post-test survey. Those who did both surveys numbered 135 which represented 45% of the population.

Reliability tests, using Cronbach alpha, gave values of 0.94, 0.90 and 0.87 for the pre-survey confidence, effectance motivation and usefulness scales respectively and 0.95, 0.91 and 0.90 for the same respective post-survey scales.

There was no significant difference between the respondents who completed both surveys and those who completed only one survey. Hence the rest of the discussion will be concerned with those students who completed both surveys. It is felt that this group is representative of the population both demographically and attitudinally.

As in most primary teaching courses there is a large number of females compared with males. Representing 86% of the 135 respondents, there are 116 females in the survey.

The group has an age distribution as shown in Figure 1. The majority of students are school leavers and a fifth are mature age students.

Figure 1. Age Distribution in Years

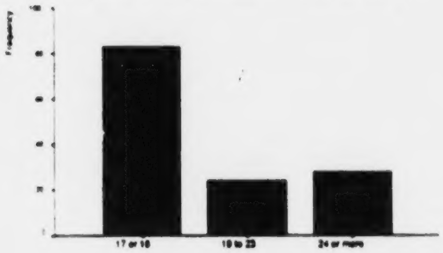
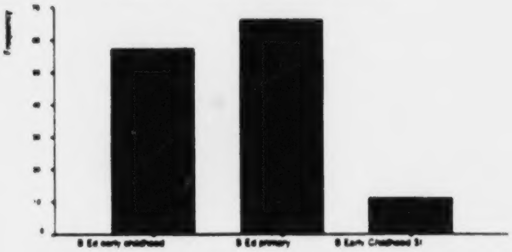


Figure 2. Degree Distribution



There are similar numbers enrolled in the Bachelor of Education (early childhood) and in Bachelor of Education (primary). A small number are studying for a Bachelor of Early Childhood Studies. The actual figures are represented in Figure 2.

Table 1. Final Grade Received

Grade	Frequency		Percentage	
HD	9	*(13)	6.7	(4.4)
A	34	(46)	25.2	(15.4)
B	55	(98)	40.7	(32.9)
C	28	(68)	20.7	(22.8)
F	9	(47)	6.7	(15.8)
Dropped	0	(26)	0.0	(8.7)
Total	135	(298)	100.	(100)

*Corresponding figures for the whole group are given in brackets.

Table 1 and Figure 3 display the semester grades of the students who responded to both surveys. The values in brackets in Table 1 give the grades and percentages for the entire *Mathematics for Teachers* group. The respondents have a higher proportion in each grade except those who failed.

Figure 3. Final Grade Received

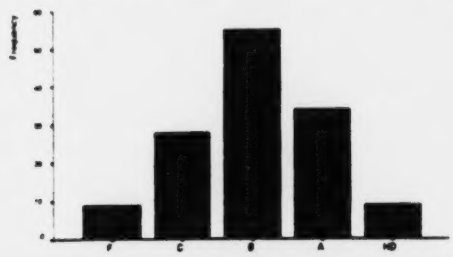
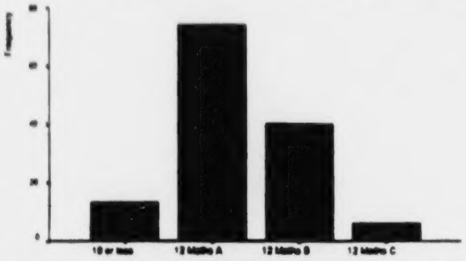


Figure 4. Highest School Mathematics Level



Most of the students have completed Year 12 Mathematics A (Figure 4). Of the thirteen who have Year 10 or less mathematics, eleven were in the 24 or over age group.

Missing data were randomly scattered throughout the scale variables and cases and deletion of all affected cases would have meant a large loss of data. The missing data were replaced by the conservative estimate of the variable mean (Tabachnik & Fidell,1989).

Table 2 shows the means and standard deviations of the total scores for each scale on the pre-survey and the post-survey and the corresponding statistics of the difference between the pre- and post-totals . There was little change.

Table 2. Statistics of the Pre- and Post- Surveys and the Difference Distributions

		Pre-Survey		Post-Survey		Difference	
Scale	N	Mean	Std.Dev.	Mean	Std.Dev.	Mean	Std.Dev.
Confidence	135	37.34	9.62	37.46	10.62	0.12	6.23
Effectance	135	37.51	8.07	37.46	10.62	-0.51	6.31
Usefulness	135	46.42	6.64	45.17	7.66	-1.25	6.10

The confidence scale shows a small, non-significant increase in confidence over the semester. However examination of the confidence distribution, Figure 5, shows an influential score. For some wonderful student his/her confidence changed by 24 points. (If only we could always have this effect!) If this score is eliminated from the sample the mean is –0.07; so virtually no change occurred. This accords with the view that confidence or self-concept is resistant to change (Way & Relich, 1993, Meyer & Koehler, 1990).

Figure 5. Difference in confidence

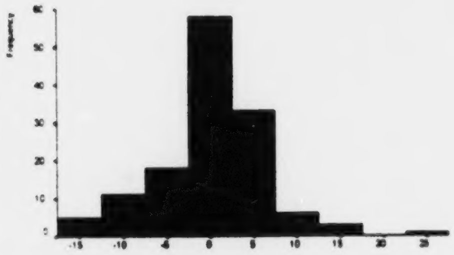
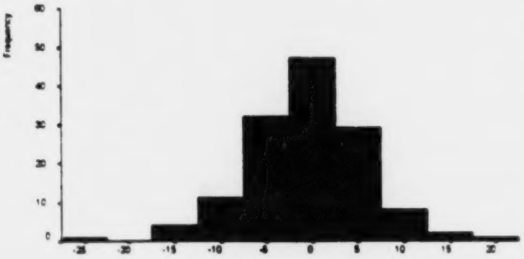
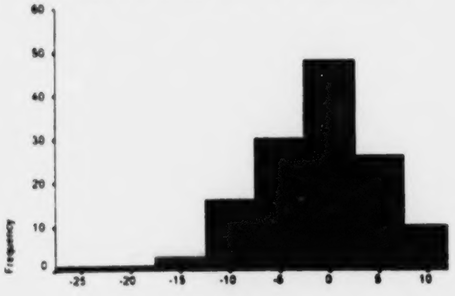


Figure 6. Difference in Effectance Motivation



The effectance motivation scale shows a decrease in attitude although it is small and non-significant. The large negative score in Figure 6. does not have much influence on the mean.

Figure 7. Difference in Usefulness



The distribution of the difference in usefulness in Figure 7. shows a negative skewness with a mean of -1.25.The sample size is large enough for the skewness not to invalidate the significance found in this difference (t = -2386, p = 0.018). This result is supportive of the usefulness findings of Carroll, (1994), Meyer & Koehler (1990), Schuck (1996) that mathematics needs to be seen as useful.

Many of the demographic variables can be used as factors in the analysis of the attitude data. Gender cannot be ignored when discussing attitudes to mathematics; for example, Fennema (1976), Leder (1992), Meyer & Koehler (1990). For this student group, however, there was no significant gender difference on any of the scales.

The scale mean differences, grouped according to age, are given in Table 3. For the two younger age groups the all of the mean differences are non-significant and negative. Though not significant, the positive means for the confidence and effectance motivation scales give encouraging results in the 24 or over group. The student who had a difference of 24 is a mature aged student and as such has inflated the confidence mean. Elimination of this score still gave a positive, but small, difference mean of 0.81.

Table 3. Scale Difference Means by Age in Years

Age Group:	17 & 18		19 to 23		24 or over	
Scale	N	Mean	N	Mean	N	Mean
Confidence	83	-0.31	24	-0.20	28	1.67
Effectance Motivation	83	-1.07	24	-1.33	28	1.86
Usefulness	83	-1.19	24	-1.97	28	-0.83

Is there a difference in the mathematical attitudes associated with the course in which the students were enrolled? Table 4 shows the scale difference means by degree course. Again there was no significant difference between the groups although usefulness shows an area of interest with $p = 0.10$. Within the groups, the B.Ed.(early childhood) usefulness data gave strong evidence of a difference in attitude ($t = -3.024$, $p = 0.004$); the difference being negative.

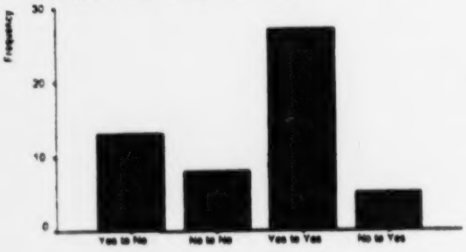
Table 4. Scale Difference Means by Degree

Degree:	B.Ed.(e.child)		B.Ed.(primary)		B.Early Child.St.	
Scale	N	Mean	N	Mean	N	Mean
Confidence	57	-0.10	66	0.33	11	0.27
Effectance motivation	57	-1.29	66	0.35	11	-1.38
Usefulness	57	-2.53	66	-0.33	11	0.09

In both surveys students were asked to indicate if they had a choice would they study the unit *Mathematics for Teachers*. As perceived usefulness influences participation and since the attitude of usefulness of mathematics has become an area of concern their change in choice could also be a reflection of these negative attitudes.

Figure 8. Change in Choice of Studying *Mathematics for Teachers*

Degree: B Ed (early childhood)



Of the 92 students in the total group who, at the beginning of the semester, indicated they would choose to study *Mathematics for Teachers*, twenty-one changed their mind from Yes to No. Forty students in the B.Ed. (early childhood), Figure 8, indicated they would study the unit in the pre-survey. Thirteen changed their mind, a proportion of 33%, in line with the change in their attitude to the usefulness of mathematics.

5 Summary and Conclusions

To measure the change in attitude to mathematics of a group of pre-service teachers while studying the unit *Mathematics for Teachers* the three Fennema-Sherman scales of *Confidence in Learning Mathematics*, *Effectance Motivation in Mathematics*, and *Usefulness of Mathematics* scale were administrated at the beginning and end of the semester.

No significant change in attitudes for the *confidence* or *effectance motivation* scales across the whole sample or within any of the three groups of gender, age and degree was found. This is supported by those studies which claim attitudes are indeed difficult to change, even within strongly supported environments (Carroll, 1994; Meyer & Koehler, 1990; Schuck, 1996). It is encouraging, however, that there is little or no drop in these scales.

It was through the negative attitude change in the *usefulness* scale that students sent a message. The group as a whole produced a small negative, but significant, difference in attitude. However the loudest message, though not unexpectedly, was sent by the Bachelor of Education (early childhood) students whose scores gave strong statistical evidence of an attitude change, albeit negative. This group also gave the highest proportion of students, who over the semester, changed their mind in their choice to study this unit.

This drop in the perception of *usefulness* is in line with the broadly held view that the mathematics taught must be seen as useful to the teachers in training (Carroll, 1994; Mayers, 1994; Schuck, 1996). But it presents the most positive challenge! It offers opportunity to address improvement in the unit which may well induce more positive results. Certainly it is the intention to try to establish what indeed those students who expressed a positive attitude about *usefulness* do consider useful, and build upon that. It is not yet clear whether the perception of *usefulness* pertains to their classroom needs or to the wider uses of mathematics in the community. Interviews should prove illuminating. So too might involving the students of the group in the research; students interviewing each other in a search for what is considered useful about mathematics in this context (Schuck, 1996). Revision of the unit *Mathematics for Teachers* can then be carried out effectively.

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A Survey of Graphics Calculator Use in Victorian Secondary Schools

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In 1997, Victoria became the first state to permit the use of graphics calculators in final external examinations. The action was seen as radical for both social and educational reasons. Concerns were raised about the propriety of using the calculators and whether their use would add to existing educational disparities. With the support of the Board of Studies, a survey of secondary schools was undertaken to gauge the response to this decision and inform further action on graphics calculator use in mathematics courses.

Introduction

Although graphics calculators have been available for about a decade, their penetration in the secondary school system has been slow - largely because of a ban on their use in the external examinations which applied universally in Australia until 1997, and which still mainly persists. A decision was taken by the Victorian Board of Studies (1995) to lift this ban from 1997 in all external examinations but a policy was adopted that the examination questions would be required to be 'calculator-neutral'. This no-advantage clause was promulgated to minimise any social inequities which might flow from the decision. In Western Australia, the decision to allow the use of graphics calculators in external examinations from 1998 was made without the use of this 'no-advantage' rule. It can be argued that this rule artificially constrains the form of the paper (Tobin, 1997) and that the desired improvements in mathematical performance are cut back. In fact Western Australia also sanctioned the use of calculators with a small degree of algebraic/calculus capability by accepting the HP-38G as an endorsed calculator model. This has not yet been permitted in Victoria.

The VCE Graphics Calculator Project was funded by the calculator companies and the Board of Studies and preceded the implementation of the Board's policy. This project was conducted in two phases – a phase for teacher familiarisation with the various graphics calculator models, run by Swinburne University of Technology and a phase for preparing sample materials using the calculators, run by the University of Melbourne (Jones, 1997; Tynan and Dowsey, 1997). Many of the educational issues of graphics calculator use in mathematics courses or examinations have been canvassed before (see Penglase and Arnold, 1996, for a review of the graphics calculator literature and Tobin, 1997, for a discussion of the impact using a graphics calculator would have made on VCE mathematics papers prior to 1997).

One of the authors (Tobin) was interested in graphics calculator penetration of the schools as a consequence of the decision and in related questions of teacher attitudes towards the Board policy and their use of the calculators in the mathematics courses. There has been little previous research undertaken on teacher attitudes although Chamblee (1995) reported that the only significant variable in quantifying teacher support for use of graphics calculators in a college algebra course was the degree of familiarity of the user. With the support of the Board of Studies, a survey was conducted in two parts - the first to determine from mathematics coordinators the level of ownership or access of calculators by students in their school and the second to determine teacher attitudes and use of the calculators in various subjects or topics.

The survey was initially sent in September 1997 with a follow up to non-responding schools in March 1998. The information to be discussed in this paper arose from the initial response to the survey.

The Survey Questionnaire

The survey was sent to approximately 485 educational providers on the basis of whether they taught the basic mathematics subject, Mathematical Methods Units 1&2. This was devised as a suitable screening procedure to ensure the survey was not limited to schools - some TAFE colleges teach VCE mathematics - but would not stray to specialised VCE providers such as language or music schools.

There were initially 165 responses from school coordinators and data from these is discussed here. The brief coordinator questionnaire included information on school size (indirectly) calculator models, booklisting policies, class sets, student access and estimations of student calculator ownership. The focus of this questionnaire was school penetration of the graphics calculators.

The more detailed teacher surveys were initially completed by over 900 teachers and some data from these is discussed here. The focus of the teacher survey was on attitudes to the Board's calculator policy and attitudes to the graphics calculators themselves. This teacher survey enables us to assess any variations in responses between school regions. types or sectors as information on these was gathered also. There are seven educational regions classified by the Ministry: Eastern Metropolitan (EM), Western Metropolitan (WM), Barwon South Western (BSW), Central Highland (CH), Gippsland (G), Goulburn-North Eastern (GNE) and Loddon-Campaspe-Mallee (LCM).

The four sectors are Government Secondary, Catholic, Independent and TAFE. The bulk of respondents in the first group were schools - only two TAFE colleges out of a possible ten providers gave any feedback. The three types of educational providers classified were boys schools, girls schools and co-educational schools.

The main target was teachers of mathematics subjects in the Victorian Certificate of Education (VCE). This spans years 11 and 12 and includes the five mathematics subjects, Mathematical Methods (MM) 1&2, General Mathematics (GM) Units 1&2, Mathematical Methods Units 3&4, Further Mathematics (FM) Units 3&4 and Specialist

Mathematics (SM) Units 3&4. It is only the unit 3&4 subjects which have external examinations affected by the new policy - these are 90 minute papers involving multiple choice, short answer and analysis questions which are called Common Assessment Tasks or CATs. Graphics Calculators and other forms of computer assistance have been encouraged in the past in the remaining external assessment, a project or problem solving task called CAT 1. The surveys also obtained information on ownership, access and use of calculators in years 7 to 10 but the figures for this are low, especially in junior levels and will not be discussed here.

Results

Coordinator Survey

Calculator Models: The Texas Instruments models dominate the market with 83% of market share. This result was expected from anecdotal evidence and appears to contrasts with the situation obtaining in other states such as Western Australia. The CASIO models cover about 15% and there are a few schools using Sharp or other calculators.

Access Policies: The information on booklisting by schools (i.e. adding a calculator to the school booklist) shows that independent non-catholic schools are much more likely to prescribe calculators than government schools. Catholic schools are also more likely to booklist calculators than their government counterparts - though to a lesser degree. This is shown in Table 1.

Conversely government and catholic schools show slightly higher rates of possessing at least one class set although this is prevalent across all sectors as seen in Table 1.

Table 1
Educational Providers: Percentage booklisting or having at least one class set

Provider	Sample Size	% Booklisting	% Class Sets
TAFE	2	100	50.0
Catholic	35	54.3	88.6
Government Secondary	85	47.7	86.4
Independent Non-catholic	40	85.0	75.0

Student Ownership: There was a wide variety of student ownership reported between school types and across year levels. Grouping the data into '95% or more' ownership and '5% or less' ownership as two extreme cases, the results for Unit 1&2 subjects (usually done in year 11) are shown in Table 2.

Table 2
Calculator ownership for subjects in Units 1&2.

Subject	Ownership (≤5%)	Ownership (>95%)
Mathematical Methods 1&2	27 %	19%
General Mathematics 1&2	40%	10%

As many students do both subjects, we may infer that students only doing General Mathematics are not yet buying the calculators much.

In Unit 3&4 subjects (usually done in year 12) the pattern shifts between FM 3&4 with its low student ownership of calculators and MM 3&4 and SM 3&4 which have higher rates. The data are shown in Table 3.

Table 3
Calculator ownership for subjects in Units 3&4.

Subject	Ownership (≤5%)	Ownership (>95%)
Further Mathematics 3&4	40%	16%
Mathematical Methods 3&4	15%	33%
Specialist Mathematics 3&4	21%	41%

Of course the students doing Specialist Mathematics would all be included in the Mathematical Methods 3&4 figure inflating that result. Students doing MM 3&4 alone or with FM could be expected to have lower rates of calculator ownership. The anomalous higher figure for low calculator ownership in SM 3&4 over MM 3&4 may be a consequence of some schools rejecting use of the calculators for more able students because of university resistance to their use later on. Some anecdotal evidence supports this.

Overall, the penetration of calculator usage has been quite high for the first year of VCE use The schools reported figures around 80% for school ownership of class sets has doubtless helped with equity issues. It could be that private ownership will not increase at a great rate if the examination embargo on 'no advantage' remains. This depends on how the teachers and students perceive the success of this policy in the 1997 papers. If the embargo is removed - perhaps progressively with Specialist Mathematics and Mathematical Methods first, there is a good base for student ownership evident in booklisting figures and a prospectively large second hand market emerging.

Teacher Survey

Board Calculator policy: In general most of teachers agree or strongly agree with the Board of Studies decision to allow graphics calculators on the VCE examinations in 1997. For each of the unit 3 & 4 subjects the pattern of response is broadly similar (see Table 4).

Table 4
Teachers support for the VBOS decision to allow graphics calculators to be used in the VCE exams 1997

	MM3&4 (%)	SM3&4 (%)	FM3&4 (%)
Strongly agree	32.3	36.5	31.5
Agree	36.3	33.2	32.8
Not sure	15.2	15.2	19.4
Disagree	9.0	7.9	9.5
Strongly Disagree	7.1	7.2	6.9

Among Further Mathematics teachers, support for the Board policy differs significantly between different types of schools (p = .01) as shown in Table 5.

Table 5
Teacher Support for calculator use in FM by school type (Agree or Strongly Agree)

	FM3&4 (%)
Boys Schools	60.4
Girls Schools	72.0
Coeducational Schools	63.2

This significant difference also occurs between different school sectors (p = .01) as shown in Table 6.

Table 6
Teacher Support for calculator use in FM for school sector (Agree or Strongly Agree)

	FM3&4 (%)
Government	61.1
Catholic	66.4
Independent non catholic	71.3

There is no significant difference between type of school or school sector in teachers' support of Board of studies decision to allow Graphics Calculators on the VCE examinations of Mathematical Methods and Specialist Mathematics. There is also no significant difference between different regions in teachers support of Board of Studies decision to allow graphics calculators in all three Mathematics subjects.

Social Equity: Despite the strong support for the VBOS policy, 73.5% of all respondents agree or strongly agree that allowing the use of graphics calculators in Year 12 examinations has added to problems of equity among students. These figures fluctuate insignificantly between types of schools, but they do vary significantly between different sectors (p =.000) and regions (p =.009).

Table 7 shows that teachers from Catholic schools are less likely to agree or strongly agree (67.8%) than their colleagues from Independent non-catholic schools (77.4%) or those from the Government schools (73.8%) that that allowing the use of graphics calculators in Year 12 examinations has added to problems of equity among students.

Table 7
Teachers attitudes to the problem of equity for each sector

	Government (%)	**Catholic (%)**	**Independent (%)**
Strongly agree	40.3	29.7	30.4
Agree	33.5	38.3	47.0
Not sure	13.4	13.7	14.3
Disagree	8.6	15.8	6.9
Strongly Disagree	4.3	2.7	1.4

Examining regional variation, Table 8 shows that teachers from Gippsland (62.1%) or Barwon South Eastern (62.8%) are less likely to agree or strongly agree than their colleagues from other regions (compare to 77.2% for Eastern Metropolitan, 75.5% for Western Metropolitan, for example).

Table 8
Teachers attitudes to the problem of equity for each location.

	(%)						
	EM	**WM**	**BSW**	**CH**	**G**	**GNE**	**LCM**
Strongly agree	38.3	37.4	29.5	38.8	29.7	51.9	22.8
Agree	38.8	38.1	33.3	30.6	32.4	25.3	52.6
Not sure	12.3	9.0	21.0	18.4	18.9	15.2	10.5
Disagree	8.7	9.0	11.4	10.2	13.5	7.6	10.5
Strongly Disagree	1.8	6.5	4.8	2.0	5.4	0.0	3.5

Future Directions: The teachers expectations of future directions show that most anticipate that the present 'no-advantage' rule will need to be replaced with a policy whereby questions can be set which exploit the power of graphics calculators more fully. Teachers were asked when they expect almost all their students to have access to graphics calculators in VCE Mathematical Methods 1&2, VCE General Mathematics, VCE Mathematical Methods 3&4, VCE Specialist Mathematics, and VCE Further Mathematics. The choice of answers was: 1997, 1998, 1999, Later, and Never. In all subjects most of respondents expected almost all students to have access to graphics calculators in 1997 or 1998, although this figures varies from subject to subject, between different types of schools within the subjects, and between different school sectors within the subjects. The overall results are given in Table 9. This issue is important for the Board in framing the next phase of graphics calculator use which removes the no-advantage rule.

Table 9

Teacher expectations on first year when almost all students will have access to graphics calculators in Year 12.

	MM3&4 (%)	SM3&4 (%)	FM3&4 (%)
1997	54.3	57.3	41.4
1998	19.7	18.7	18.3
1999	13.6	13.3	16.6
Later	9.4	7.1	15.9
Never	3.1	3.6	7.8

For all subjects, teachers' expectations that almost all students will have access to graphics calculators in 1997 or 1998 were significantly lower for teachers from Government schools than for teachers from Independent non catholic schools. Therefore, if we consider over representation of Independent non catholic schools in the survey and under representation of Government schools, we must conclude that overall expectations that almost all students will have access to graphics calculators in 1997 or 1998 are higher in the survey than it must have been in reality. The additional survey data currently being analysed has ensured a more equitable response from the government sector.

Gender Differences: For all subjects, teachers' expectations that almost all students will have access to graphics calculators in 1997 or 1998 are significantly higher for teachers from Girls' schools than for teachers from Boys' and Co-educational schools. In fact the survey of teachers reveals greater interest in the graphics calculators by Girls schools as indicated by Table 10. In this Table we list the percentages of respondents of each type for which 90% or more of students have their own graphics calculator.

Table 10
Calculator Ownership in Year 12 mathematics by Type.

	MM3&4 (%)	SM3&4 (%)	FM3&4 (%)
Boys Schools	33	42	7
Girls Schools	87	87	43
Co-educational Schools	38	43	16

Summary

The results of this survey suggest strong teacher support for the Board's policy to introduce graphics calculators despite a prevailing view that there is inevitably some social disadvantage at least in the transition period. The survey sample size is substantial for both coordinators and teachers and in all school types (except TAFE). sectors and educational regions but the results still need to be treated with some caution as the sample is to a degree self-selecting and there was a higher response from independent schools than they represent in the educational system. The graphics calculator penetration has been substantial in VCE year levels – particularly in Mathematical Methods 3&4 and Specialist Mathematics 3&4. Predictably, as yet there has been less use of the calculator in middle or junior school levels. The survey is currently in the second phase of analysis but some interesting results are already apparent. Although the strong uptake by independent schools could be anticipated, the striking variation between how girls schools and the other types have responded in adopting graphics calculators is surprising. There has been some evidence in the past that the use of graphics calculators may favour girls (see, for example, Ruthven, 1990; Boers and Jones, 1994). This issue will need further research.

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The Search for Pattern: Student Understanding of the Table of Values Representation for Function

Julie Ryan and Julian Williams

This study explores student understanding of the table of values representation for the function concept. One hundred and seventy eight students in Years 8, 9 and 10 across three schools in Melbourne undertook two pen-and-paper tests which sought to uncover the nature of the links students made between different representational forms for function. For the table of values representation there are many students who are operating in an arithmetic or pre-algebraic frame across the three year levels. There are low levels of response for algebraic patterning and indications that most students are at a rhetorical or syncopated stage of symbolism.

The Function Concept: Understanding of Symbols and Structure

The function concept is a crucial one in secondary school mathematics as it can be seen as a unifying idea throughout the algebra curriculum. The function concept ultimately requires the student to understand *structure*. This conceptual understanding requires a significant step from numerical processing and it is the bridge between arithmetic and algebra that is a focus of different introductory perspectives to school algebra. The four representations for function promoted in current curricula involve natural language description, symbolic, graph, and table of values representations. These representations and translations between them were the focus of a larger study, but only student understanding of the table of values representation is reported here.

Through the use of *symbols* in algebra a dexterity of thought becomes possible, where mental objects can be manipulated, or can be held suspended and unresolved, at will. The compactness of symbol for idea in algebra is similar to word symbols in language but the special syntax and grammar of algebra are not always intuitive and thus present a cognitive leap for the student (Herscovics, 1989; Kaput, 1989). Symbolisation requires formal levels of thought and the encapsulation or *reification of processes as objects* requires a major step forward from arithmetic thought (Harel & Dubinsky, 1992).

In moving from arithmetic to algebra the student needs to transform implicit knowledge of rules and structure into explicit knowledge. Making an idea explicit (or explicitation in van Hiele's terms) requires abstraction, a use of symbol to represent classes of objects, and a new syntax of expression. This may be difficult for students.

From Arithmetic to Algebraic Thinking

An idea and its symbol are subject to separation. Such detachment is a powerful achievement of symbolic algebra, yet it is a trap for the learner. That the ubiquitous x has *flexible meaning* causes confusion too for the beginner. Is it a number, any number, a set of numbers, or all numbers? Can we work with it when we do not know its value? Once x becomes a conceptual entity in itself, detached from known number, the learner has algebraic power of compression of representation, and can manipulate the mental and written objects according to accepted algebraic rules, and can then return and reattach meaning within the context of a problem.

The flexibility of thought required here is considerably greater than in arithmetic, which has been the grounding for the beginning algebra student. The arithmetic frame is usually strongly established and the imperatives of student processing behaviour in arithmetic are different in nature to that required in algebra. To think algebraically is distinctly different to thinking arithmetically, yet in the transition to algebraic thinking use of the arithmetic frame and generalisation from arithmetic behaviour is sensible.

Generalisation is seen to be the defining characteristic in the shift from an arithmetic perspective to an algebraic perspective: *generalising numerical patterns* is at the heart of early algebra (Sfard, 1995). Consideration of numerical patterning in arithmetic and geometry are often seen as an intuitive root for the beginning algebra student (Bell, 1988;

Mason, 1996) and current curriculum statements are drawn to them as suitable *starting points* (Australian Education Council, 1991; National Council of Teachers of Mathematics, 1989). A notion of variability is also a significant stepping stone essential for algebraic understanding. These two concepts, generality and variability, embody the structural perspective that is the essence of algebra.

The function concept draws together the key concepts of variability and generality. The study of relations between variables is a prime focus of algebra. The dependency relationship can be demonstrated in different ways: expressed *compactly* by a symbolic rule or formula, *particularised* in a numerical table of values, or *visualised* in a graph.

The Search for Pattern: Generality and Variability

While the Australian curriculum document *A National Statement on Mathematics for Australian Schools* (AEC, 1991) states that "(m)athematics brings to the study of patterns an efficient and powerful notation for representing generality and variability, and for reducing complexity – algebra" (p. 187), the table of values representation for function is not promoted *explicitly* in any of the sub-strands as a medium for understanding the basic concepts of algebra, namely generality and variability.

The different representations for the function concept listed in the AEC document each have their own structure and therefore require a different perspective of the student. This study sought broadly to answer the questions: (1) What is the nature of student understanding of the function concept? (2) What is the nature of student understanding of each of the representations for function? (3) What connections are made across the representations? (4) What are the implications of the findings for teaching?

With these broad questions in mind the role of verbal description (so-called natural language) and understanding of the tabular representation for function were seen to be of particular interest in the light of limited related research. Calls for building on student numerical patterning skills for introductory algebra (AEC, 1991) suggested that an examination of that ability and its relation to algebraic development was required.

Natural Language, Table of Values and Symbolic Algebra

In order to capture the underlying function rule for a table of values a student may use a natural language description, for example 'double the number and add one' or 'add four to the number'. In attempting to move the student to *symbolic* language the classroom teacher may temporarily use syncopated algebra where symbols *abbreviate* the inputs and outputs and the mapping is made more explicit: 'the output is four more than the input' to 'y is 4 more than x' to 'y is x plus 4'. Thus the relationship of input-output is moved (subtly) from comparison to process. The *syncopated algebra*, 'y is x plus 4', is sometimes used as a transitional language in the development of symbolic algebra '$y = x + 4$'. (From this point in the discussion 'natural language' will refer to this syncopated algebra common to classroom dialogue in the introductory stages of algebra.)

The historical role of rhetorical algebra (prose) and, more particularly, syncopated algebra in the evolution of symbolic algebra (Harper, 1987) demonstrated that there were significant mental stumbling blocks in moving from the cumbersome prose form of description to a fully *symbolised* algebra where new mental objects serve to compress both context and operation (Kleiner, 1989).

The table of values is less flexible and compact form than the symbolic form, yet its structure and syntax, too, need to be understood. It does not capture generality, but lists *examples* of numerical values paired under the functional relationship. The organisation and relationships of the parts of the table and its structure are different again to both natural language and symbolic algebra. The student is required to identify a set of x, a set of y, and to decipher an ordered set of arithmetical operations between a pair of x and y which holds for all x and y pairs. The generality is implied in that the relationship should also hold for extrapolation and interpolation under a relevant domain set.

For algebraic understanding of $y = x + 4$ the student must merge both structural conception and process into a conceptual entity – an algebraic procept – and be able to think flexibly so that either concept or process can be invoked as required (Gray & Tall, 1994): the duality of the function concept must be managed as a dichotomy of relationship and process *at will*. The student requires flexibility of perspective. Similarly, for the table of values the student needs to develop a proceptual view of the table as both relationship and process.

Research Methodology and the Test Items

For this study two written tests were formulated to investigate student understanding of key sub-concepts of the function concept across the four representations. The first test focussed on the *table of values representation* for function and the second test focussed on the *graphical representation* for function. The overall context for both tests was the linear function. The year levels chosen for study were years 8, 9, and 10 in secondary schools. The tests were undertaken at the end of the school year in October, November and December, and the age range of the students was 13 to 16 years of age. A sample of 178 students across years 8, 9, and 10 in three schools was considered to be appropriate to capture a representation of the most common responses to the test items. Each of 9 class groups was visited twice in their own school environment and each class was asked to complete two written tests each of one hour duration over a period of 4 to 6 weeks.The schools in the sample came from the government school sector in the Australian State of Victoria and had quite different profiles in terms of students and academic resources. This sample was designed to provide a variety of responses from a broad spectrum of students' progress in understanding the function concept. The three participating schools provided a broad sample of Victorian students in terms of background, achievement and opportunity.The test items discussed here were designed to reveal students' understanding of functions represented in tables. Students were asked to state a relationship or rule between the numbers in the x and y columns of the given tables, firstly in natural language (Da), then in algebraic form (Db), and lastly to supply y values for given x values (Dc). These items were designed to assess: (1) Which structural features of a mathematical table of values are recognised by students? (2) What patterns are seen by students and which ones are debilitating in attaining an algebraic view? (3) Whether some students are in a rhetorical stage of symbolism, (4) Which rules present a more difficult translational task, and (5) What are the error phenomena?

D4

x	y
1	3
2	5
3	7
5	11

(a) Describe the relationship or rule between the numbers in the x and y columns in words:

(b) The rule is y =

(c) If these also belonged to the table, fill in the correct y values to match the x values.

x	y
-1	
0	
10	

Figure 1. Items Da4, Db4, Dc4

The sequence of linear functions in section D was: (D1) $y = x + 3$, (D2) $y = 4x$, (D3) $y = x$, (D4) $y = 2x + 1$, (D5) $y = 5 - x$, and (D6) $y = x - 5$. Items D1, D2, D3 and D6 are uni-operational functions in the sense that one arithmetical operation is performed on the independent variable, and numbers D4 and D5 are bi-operational in the sense that two operations are performed on the independent variable. The six linear functions for Da, Db and Dc were chosen to represent suspected different levels of difficulty. The first four questions (D1 to D4) all had $x = 1, 2, 3, 5$ in the initial table with their matching y values. The omission of $x = 4$ was meant to break the pattern of x-change in case a vertical cue was being used by the student. The last two questions (D5 and D6) had a different sequence of the independent variable values: D5 had $x = 1, 2, 3, 4$ to monitor the effect of the distracting pattern in the y column and D6 had $x = 7, 4, 2, 1$ to monitor the effect of a decreasing change in the x values.

Analysis

In the analysis of student responses a tree diagram was used to show the connections and stumbling blocks between the three tasks of each item of section D. The numbers in the tree diagram below (Figure 2) represent the numbers of students correct (c) or not correct (nc) for each task of D4. Thus the left-most path in the tree diagram labelled as path (1,1,1) represents those students who successfully responded to all tasks (verbal - Da4, algebraic - Db4 and numeric - Dc4). The next path in the sieve, labelled as (1,1,0), represents those students who were correct on both Da4 and Db4 but not correct on Dc4.

One interesting path through different patterning abilities is detailed by arrows in the figure. This path, described as path (0,0,1), represents those students who were unsuccessful in both Da4 (verbal description) and Db4 (symbolic algebraic description), but were successful in Dc4 (numeric patterning). These students perhaps had some strategy other than verbal description or algebraic description which allowed them to respond correctly on the final task. The path is characterised as *'successful inarticulate patterning'*.

The eight paths are characterised as: (1,1,1) successful articulate algebraic patterning; (1,1,0) unsuccessful articulate algebraic patterning; (1,0,1) successful rhetorical patterning; (1,0,0) unsuccessful rhetorical patterning; (0,1,1) successful articulate algebraic patterning; (0,1,0) unsuccessful articulate algebraic patterning; (0,0, 1) successful inarticulate patterning; (0,0,0) non-patterning. (Path (0,1,1) is characterised in the same way as (1,1,1): successful articulate algebraic patterning. Similarly, path (0,1,0) is characterised in the same way as (1,1,0): unsuccessful articulate patterning. The assumption is made that natural language description ability is subsumed in algebraic description ability.)

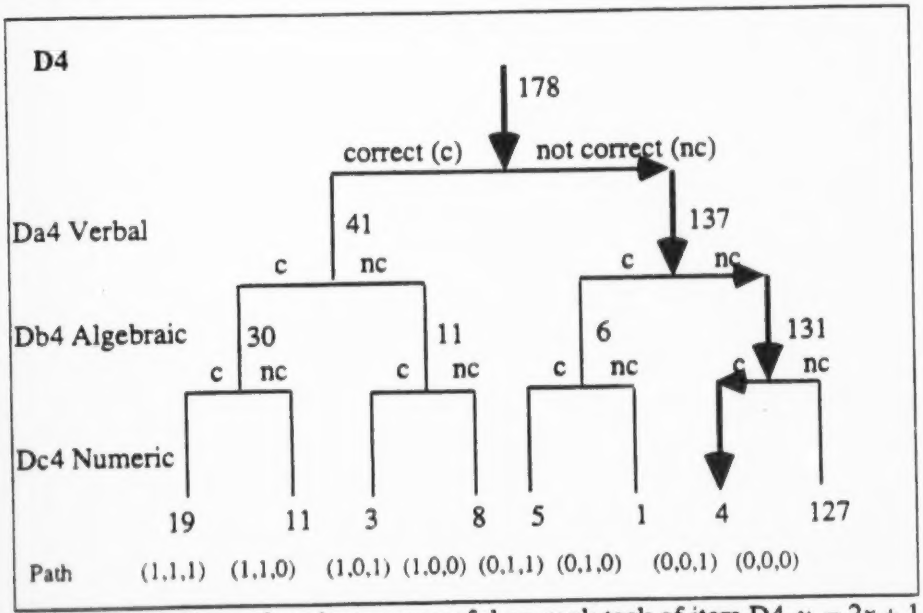


Figure 2. Numbers of students successful on each task of item D4, $y = 2x + 1$

In all six questions it appears that there are students who can complete a table of values and yet are not able to correctly express the underlying function in natural language or symbolic language. Their path is characterised above as 'successful inarticulate patterning' (0,0,1). For this group the pattern within the table is accessible in some other way. Table 1 below indicates the number of students in this category for the different table functions.

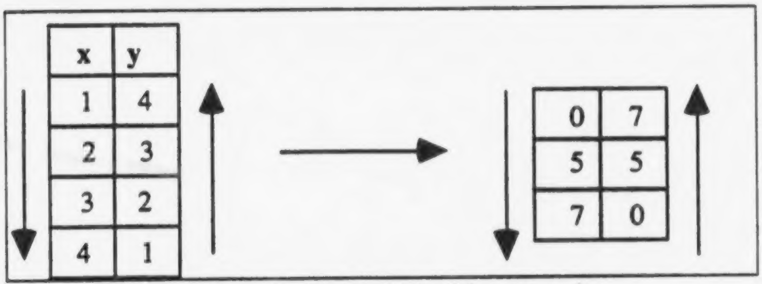
Item	Table function	Number of students (n = 178)
D1	$y = x + 3$	22
D2	$y = 4x$	10
D3	$y = x$	15
D4	$y = 2x + 1$	4
D5	$y = 5 - x$	19
D6	$y = x - 5$	12

Table 1. 'Successful inarticulate patterning' for different functions

The three tasks in each item were used to form three scales and a reliability analysis was undertaken on SPSS (no. of items = 6, no. of cases = 178) for each of the scales. Statistical analysis shows that there is a strong positive correlation for each item with Scale Da (except for item Da5 where there is a weak positive correlation). That is the items contribute reliably to the scale measure Da of 'verbal patterning ability' (Cronbach-alpha = 0.777). There is a strong positive correlation for each item with Scale Db (except for item Db5 where there is a weak positive correlation). That is the items contribute reliably to the scale measure Db of 'algebraic patterning ability' (Cronbach-alpha = 0.851). There is a strong positive correlation for each item with Scale Dc (except for item Dc5 where there is a weak positive correlation). That is the items contribute reliably to the scale measure Dc of 'numeric patterning ability' (Cronbach-alpha = 0.786).

The D5 items had the weakest positive correlations with each of the three scales. The table function for these items was $y = 5 - x$. This function can be seen to be different from the others in its presentation: the given table is shown in Figure 3 below. The entries as given proved to be overwhelming distracters from the (horizontal) function pattern where students were drawn rather to the vertical patterns in the x and y columns.

The down/up *parallel patterning* distracter in item Dc5 resulted in 12.4% of students responding as shown in Figure 3 below for $y = 5 - x$. This response is a non-algebraic mapping where students have disregarded the concept of correspondence between variables.



x	y
1	4
2	3
3	2
4	1

0	7
5	5
7	0

Figure 3. Item Dc5: Parallel table patterning

Facility and Error Types

Verbal patterning: As shown in Table 2 below the functions in order of ease of *verbal description* were $y = x$, $y = x + 3$, $y = 4x$, $y = x - 5$, $y = 2x + 1$, $y = 5 - x$. For translation from table to verbal description, the identity function is the easiest, and the additive function is easier than the multiplicative function. The uni-operational functions are easier than the bi-operational functions.

Item	Function	Facility (%)	Error types
Da1	$y = x + 3$	60.7	reversal
Da2	$y = 4x$	46.6	reversal
Da3	$y = x$	62.9	
Da4	$y = 2x + 1$	23.0	horizontal additive patterning
Da5	$y = 5 - x$	2.8	down/up parallel patterning
Da6	$y = x - 5$	34.3	reversal

Table 2. Facilities and error types for 'verbal patterning ability'

The identity function (Da3) was variously described correctly as (y is) 'the same' (as x) or 'one times', the additive function was described correctly as 'add 3', the multiplicative function mostly described as y is '4 times' x but a small number of students used a (correct) notion of double add double, that is $y = 2x + 2x$. The subtractive function was mostly described as 'take away 5'. The much harder bi-operational function, $y = 2x + 1$, (facility 23%) was described mostly as 'double and add 1', but it was susceptible to incorrect *horizontal additive patterning*. The 12.4% of students who were drawn to this patterning described the relationship or rule between the numbers in the x and y columns in terms of a sequence of additions between each of the four pairs of (x, y) given, that is 'add 2, add 3, add 4, add 6'. They were unable to translate this to either algebraic rule or apply it correctly to the table items in Dc4. Thus a dependence on an additive perspective of table patterning constitutes a cognitive obstacle.

The most difficult function to describe verbally was $y = 5 - x$ (facility 2.8%). The function is bi-operational in terms of 'add 5 to the negative of x', however it was expected that most students would describe the relationship as 'x plus y is 5'. This was not the case. As discussed above there was a powerful distracter at work. 27.0% of students (across all year levels) who were drawn to the vertical *down-up parallel pattern* of the x and y columns.

Algebraic patterning: There was also a very small number of students (n = 6) who *reversed* the relationships. This was pertinent to items Da1, Da2, and Da6. The other type of error for verbal items was reference to the vertical patterning of the table in terms of the incremental additive change in x and y. Apart from the 48 students (27%) drawn to this patterning for $y = 5 - x$ discussed above, there were 11 students who used the same strategy on other tables as well. Of these 11 students, 5 of them were only correct in the identification of the identity function and the additive function, but incorrect for all other tables, while the other 6 were incorrect on all other items. Their usage of this *vertical additive patterning* then can be seen to be persistent and debilitating.

The items of Db(1-6) constituted a reliable scale measure of 'algebraic patterning ability' as indicated above. As shown in Table 3 below the functions in order of ease of algebraic description is the same order as for 'verbal patterning ability'. The most difficult function to describe algebraically was $y = 5 - x$. This item had a dramatically poor facility of 2.3% with most students omitting it. As discussed in the previous section relating to verbal patterning the problem lay in the distraction of the parallel reverse patterning in the x and y columns.

Item	Function	Facility (%)	Error types
Db1	$y = x + 3$	39.3	verbal, process
Db2	$y = 4x$	36.0	verbal, process
Db3	$y = x$	47.2	verbal, process
Db4	$y = 2x + 1$	20.2	verbal, single pair
Db5	$y = 5 - x$	2.3	verbal, process
Db6	$y = x - 5$	27.5	verbal, process

Table 3. Facilities and error types for algebraic patterning ability

The largest numbers of incorrect responses were from students who gave a verbal rule without algebraic symbolism. The next largest group attempted algebraic symbolism but failed to include the independent variable in the statement. This group concentrated on the *process* or action. For $y = x + 3$, 12.9% of students wrote $y = +3$, or $y = 3$ times larger; for $y = 4x$, 7.3% of students wrote $y =$ multiply by 4 or $y = 4$; for $y = x$, 8.4% of students wrote $y = 0$, or $y = +0$, or $y = 1$, or $y = \times 1$; for $y = 5 - x$, 2.8% of students wrote $y = 5$; and for $y = x - 5$, 5.6% wrote $y = 5$ or -5.

A small number of students (5.1%) drew algebraic patterning from *a single pair* of data for the bi-operational function $y = 2x + 1$. Here the algebraic rule was given as $y = x + 2$, or $y = +2$, or $y = 2$. Students here noticed that the elements of the first pair in the table (1, 3) had a difference of 2. The numbers of reversals in algebraic patterning were minimal.

Numeric patterning: The items of Dc(1-6) constituted a reliable scale measure of 'numeric patterning ability' as indicated. As shown in Table 4 below the functions in order of ease of verbal description is almost the same order as for verbal and patterning abilities with Dc1 and Dc3 interchanged. However, the facilities for numeric patterning are higher than for the previous abilities except for question D4.

Item	Function	Facility (%)	Error types
Dc1	$y = x + 3$	74.2	
Dc2	$y = 4x$	49.4	arithmetic structure mapping, zero, -1
Dc3	$y = x$	70.8	
Dc4	$y = 2x + 1$	17.4	arithmetic structure mapping, zero, -1
Dc5	$y = 5 - x$	12.9	arithmetic structure mapping, down/up parallel patterning
Dc6	$y = x - 5$	41.6	arithmetic structure mapping

Table 4. Facilities and error types for 'numeric patterning ability'

The responses here uncovered further detail of students' patterning perspectives as some students mapped an 'arithmetic structure' from the initial table to the new entries rather than the function structure. For example, for $y = 2x + 1$ some students (11.2%) extracted the table pattern as +2, +3, +5 from the first three entries, and mapped this on to the new entries giving (-1, 1), (0, 3), and (10, 14), or extracted the pattern as +2 from the first pair to give (-1, 1), (0, 2), and (10, 12). These students have failed to recognised the underlying algebraic structure of the table and have taken no account of the *order* of the entries or correspondence of the *variables*.

Conclusions

The research questions considered the connections students make between the table, natural language and symbolic algebra. The analysis here indicated that many students did not recognise the structural features of the mathematical table of values for the linear functions used here. What were termed as 'an arithmetic structure mapping', 'a vertical additive patterning' and 'a horizontal additive patterning' indicated a non-recognition that two sets of values of corresponding variables were *co-related* in the same way – that there was a generality of *correspondence*. Students who responded with these perspectives can be seen to be responding pre-algebraically.

The parallel patterning behaviour in D5 dramatically showed that other patterns in tables are a distraction if an algebraic view is not strongly established. Similarly a vertical additive patterning which paid attention to incremental change on x and y without recognition of correspondence was demonstrated as persistent and debilitating.

A trace of the various paths for successful and unsuccessful translations between the three tasks showed that there were substantial numbers of students who were not able to supply algebraic symbolism but who were strong on verbal and numeric patterning ability. These students came from all the year levels. Their responses were termed 'successful rhetorical patterning'. Their responses on the algebraic tasks were sometimes partially symbolised thus constituting a syncopated stage of symbolism.

There were students who were able to supply the numeric values but who were unable to express the pattern of the table verbally or algebraically. Their responses were termed 'successful inarticulate patterning' and students exhibiting this response were found to be persistent in this response type. There were also large numbers of responses termed 'non-patterning' indicating that for many students the function concept is not established at all. Reversal errors occurred in all translations but were most common in the verbal description of the table pattern. There were problems with processing zero and negative one in the numeric supply tasks. Arithmetic perspectives for table patterning were indicated in the three translations in drawing the table pattern from one pair of data, from vertical patterning alone, or from a sequence of operations. These errors can be seen to arise from an action or process conception of function (Harel & Dubinsky, 1992).

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Parent Newsletters supporting Mathematics in the Junior Primary School

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This paper presents research into the effect newsletters had on parental perceptions of junior school mathematics. Information on classroom content allowed parents to recognise and individualise mathematics that occurred spontaneously in the home. Interviews of parents and teachers revealed discrepancies in the way mathematics was viewed. All the teachers and participating parents were positive about the benefits of the newsletters.

Introduction

Parental involvement

Research overwhelmingly endorses the benefits of increasing parental involvement. Ramsay, Harold, Hawk, Marriott and Poskitt (1992) summarise a review of international research: *there are major gains for parents, teachers and above all else students if caregivers, teachers and the school community can enter into a meaningful relationship in terms of the children's education* (p.168). Wylie and Smith's (1995) study of children's progress through the first three years of school in New Zealand gives an insight into why the children of high-contact parents are advantaged. They found that when children encountered difficulties teachers tended to put the onus on themselves to help the child or they limited their expectations, but

> *If there is regular informal contact with parents, and parents are seen as receptive by teachers, then such ordinary "humps" in learning paths can be discussed, and worked on co-operatively ... But if the contact is irregular, then it is much harder to talk of a child's progress, and in talking between teacher and parent, alight upon the (often) small things which cumulatively can make a difference.* (p.95)

However large numbers of parents do not have regular contact with the teacher of their child (Epstein, 1987). If involvement is to be encouraged parents need to be given information about the school and its programmes (Ramsay et al., 1989).

Junior School Mathematics

New Zealand studies have revealed that parents have little knowledge of mathematics programmes in junior primary classrooms (Wylie & Smith, 1995). Visser and Walker (1993) found that nearly two thirds of parents did not know enough about their child's mathematics programme to comment on it. They recommended that newsletters be used to improve this. Cooper, Whitehead and Patrick's (1996) study of parent education in relation to mathematics found a regular written newsletter was parents' favoured method of receiving information.

Newsletters may also address the apparent difficulty children experience with transferring skills taught in the classroom to mathematics problems encountered in everyday living (Carraher, Carraher & Schliemann, 1985; Lave, 1988; McIntosh, Reys & Reys, 1992). Mathematics is sometimes so well embedded in everyday activities that it lacks visibility (Carr, 1992). Drawing parents' attention to the mathematical nature of routine household tasks may increase their awareness and result in a higher incidence of explicit role modelling. This approach is used in a successful mathematics intervention programme at the new entrant level (Young-Loveridge, 1993).

The curriculum document for New Zealand schools stresses the need to teach mathematics *"in a meaningful context"* but researchers have documented how difficult it is for a classroom teacher of junior school children to do this effectively (Desforges & Cockburn, 1987). Tizard and Hughes (1984) identified three advantages parents have over teachers aiming to teach *"in a meaningful context"*: the extensive range of activities that take place within the home; the embedded nature of the learning; and the common memories shared by parent and child. Parents' ability to use these advantages was examined in this research, which investigated the effect newsletters had on parents' ability to support their child's mathematics education.

Method

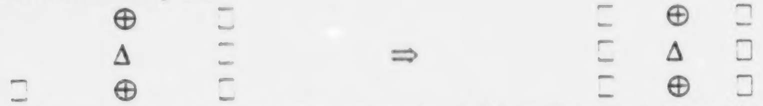
An action research methodology was used to trial and improve a regular newsletter. The newsletter aimed to increase parental awareness of mathematics in the Junior Primary School.

Figure 1: Sample mathematics newsletter used at Milland School.

Maths News

This week we've been copying patterns and using these words to describe positions: "first", "second", "third". For instance:

- Our teacher gave us cards with patterns drawn on them. We had to use bottletops, rods, counters and cubes to finish the pattern.



- In pairs we took turns at making our own alternating patterns and then asking our partner to carry them on.

⊕▯⊕▯⊕ ⇒ ⊕▯⊕▯⊕▯⊕▯⊕

our pattern — our partner carries it on ....

- We drew pictures of ourselves. The teacher lined up our pictures, telling us "Kahu is first, Jan is second, Maia is third" We had to line up in the same way and then checked our pictures to see if we were correct.

If you would like to support our classroom mathematics with activities at home you could help us look for patterns around us and ask us to describe them. If you're looking for them you'll probably see quite a few but to get you started we've made a list:

- left, right, left, right - an alternating pattern we make as we walk down the street.
- "ring-ring - *pause* -ring-ring- *pause*- ring-ring" - the pattern of sound from the telephone
- using blocks, buttons, bottletops or cotton reels to make unfinished pattern that we can complete. We could take turns at this with you.
- arranging our dinner vegetables or dessert in a pattern
- talking about what comes next. eg Jinny has opened the door, I've closed it, Alfie has opened the door, now what will happen next?

Have fun. Happy Mathematics!

Sampling

Initially five classes (from New Entrants to Year 2) trialed newsletters at Milland, a large primary school located in a military housing area. Feedback from parents and teachers was used to modify the format and content for two subsequent cycles in a single Year 2/3 composite class at Pallace, a small urban school. Both schools were classed as decile eight on a SES scale of one to ten.

All participating parents were sent questionnaires and invited to individual interviews. This self-selection lead to homogenous interview group, whom the teachers categorised as "high-contact parents". Five mothers and three teachers were interviewed at Milland. The parents of six children (six mothers and four fathers) and the teacher were all interviewed twice at Pallace. All interviews were semi-structured and took place in parents' homes. The rapport established by the interviewer was demonstrated by the eagerness with which parents discussed a wide range of issues relating to their child's mathematics and education in general.

Newsletters

Figure 1 shows a sample newsletter from Milland school. These were produced to match the activities in the Beginning School Mathematics (BSM) resource used in New Zealand schools. They were sent home approximately once a week, as children finished the activities. Subsequent newsletters at Pallace were longer but sent home approximately three weekly, in accordance with the topic-based programme which operated in that classroom.

Results

Mathematics - the forgotten subject

Before the newsletter parents reported that unless the parent had helped in the classroom during the day, they were unaware of the content of school mathematics lessons. Natasha's mother summed up the situation:

> *I presume that Natasha is doing fine in her maths but I don't actually know. You never really know. They bring their reading home so you know where they are with that. You tend to get an idea of what they're doing in ...* [language, handwriting, physical education, art and music]
>
> *Mathematics is the big mystery one really. I had no idea of what they were doing and I don't really know how well she is going. I never think to ask her what she's doing in maths and she never tells me anything that she's doing. In fact it could be the forgotten subject ...*

This ignorance of the content or their child's progress in mathematics was mentioned by almost all the interviewed parents. The exceptions were Jack's mother, a teacher aide who occasionally worked in her son's class, and Joe's mother who taught junior classes at another school. However, even Joe's mother commented that the newsletter "*served as a good reminder that every now and then these are the sort of things that I need to be talking about. Remembering when we go to the supermarket to talk about money and how many and the sizes...*"

Parents remarked that, before the research interviews, they had been unaware of their relative ignorance of mathematics content and their child's progress. It was only the interviews that had caused them to examine their knowledge of the mathematics covered in the classroom in relation to the other subjects. Three of the Milland mothers and one Pallace mother could not recall mathematics ever being mentioned at parent interviews. Milland School teachers agreed with the parents' perception of the situation.

Hannah, one of the Milland teachers, noted that as a result of the newsletters she *"got 20 out of 24 parents* [attending the parent-teacher interviews] *and probably 50 percent of them asked about maths and that's unusual. They don't usually ask about maths."*

The "code of silence" on the subject of mathematics was also reflected in the documentation Milland School gave to parents. No mention of mathematics is made in the "Beginning School" handbook which includes pages on reading, writing and becoming independent. Until the interviews, the teachers had been unaware that mathematics had been overlooked. Teachers reported that mathematics "*took up too much room*" to be included in the weekly homework sheets.

Parents' actions in the absence of information

Despite mathematics being *the forgotten subject,* parents felt that it was a very important component of their child's education. All of the interviewed parents had purchased books, magnetic numbers or games to support their child's mathematics. In the case of the Year 1 and 2 children at Pallace, all the parents had undertaken some form of mathematics coaching: two showed the interviewer books of arithmetic facts, two had games advertised as increasing a child's mathematics ability, and three had drilled their children on multiplication facts. In some families this support for rote learning of arithmetic began in the kindergarten years.

> *Leah has always done sums. Since she started school. It used to be quite a game to her. She used to do adding. She could add up to about 10+10. She would just look up at the ceiling and the next minute the answer would pop out.*
>
> [Jack learns times tables] *how I learnt it. We used to start with "two ones are two, two twos are four..." calling out loud. And we talked about it because I think at that stage I was in his class as well and I heard them talking about two groups of three and two groups of four so we talked about it like that.*

The emphasis that parents placed on early mastery of arithmetic facts is in direct contrast to the achievement objectives in the New Zealand Mathematics Curriculum which do not require recall of addition/subtraction facts until Level Two (approx Year 2-Year 6) and multiplication at Level Three (Year 4-Year 8).

Given the lack of information parents received about mathematics it is not surprizing that wide discrepancies existed between the views of junior mathematics held by teachers and parents. A Milland teacher explained that parents *"think of maths as numbers. And when they don't hit numbers straight away they get knocked. I regularly have parents say, "When are they going to have a book?" or "When are they going to start adding up?"* Many of the parents confirmed this view by mentioning their surprise at the "fun" nature of the mathematics they read about in the newsletter. Given that the interviewed parents were all high-contact, it could be expected that other parents are also ignorant of the decreased emphasis on rote learning of arithmetic skills.

Parents' response to the newsletters

Questionnaire responses indicated that parents liked the newsletters. They liked knowing "what was going on".

> *I asked about mathematics* [before the newsletter] *but she said "We are not doing any proper maths. We are just doing kindergarten*

> *play". So when I got the newsletters, that made me aware that they did some sort of maths, even if at first she didn't realize it. It had a calming effect for myself.* (Helga's mother)

For Joe's mother, who is a teacher herself, the newsletter *"served as a reminder that every now and then these are the things I need to be talking about".* For most of the other parents the content was surprizing: *"I never thought of doing anything like that before. I mean I would have done it when I was at school but I certainly didn't think of doing it"* (Donald's mother).

All of the interviewed parents wanted the newsletters to continue. Two recalled past experiences of themselves as students bringing home mathematics in Secondary School. *"My Mum couldn't help me once I got to College - just didn't even want to try. If we got these* [newsletters] *right through, then we'd be learning with our children and we'd never have that sort of problem"* (Rachael's mother).

Nineteen of the 22 parents responding to the questionnaire said they tried *Maths News* activities. Most had adapted activities to fit their resources and the personal interests of their children (including younger siblings). Natasha's mother talked about dinosaurs; Tony's parents counted deciduous trees in twos and fives; Jack's mother talked with her daughter about cutting sandwiches into quarters and halves. This flexibility appeared to result from being aware of what was being studied in the classroom. Perhaps the best example was given by Linda's parents who searched their house on a spontaneous "Roman Numeral Hunt"; finding a watch, chapter headings in books and TV programmes that featured roman numerals. Linda then gave her mother a formal lesson on the value of Roman Numerals - and even devised a test which was sat (and passed) by both parents!

Newsletters affected the way mathematics was viewed at home

Parents reported that the newsletters had made mathematics in their daily routines more visible:

> *"... things like bigger than, smaller than. You don't realize how much you talk to your children about maths until it is brought to your attention .. "* (Liz's Mother)

Mother: *It just makes your everyday thinking more aware. Like setting the table: "If two of us are going to be away tonight then how many knives will we need?" and things like that because when you are not mathematically minded it is helpful to have reminders about what you could be doing.*

Int: *Things you do anyway?*

Father: *Yes. We've just been having* [a mathematics conversation] *a few seconds ago. What is the temperature tomorrow? 16 degrees ... That sort of thing we do all the time. But I must admit that this is only since ....*

Mother: *The newsletters sort of made us more aware..*

Father: *The newsletters probably started us off down that track.*

(Adele's parents)

An additional bonus of the newsletter was the flow-on effect to siblings. Six of the children's parents reported that the newsletters were useful for younger children.

"... *our four year old as well. He was busy eating "one quarter of my toast". That was exactly what he said!"* (Jack's mother)

> [It is] *so easy to teach them if you know that they will be doing that at school. I'm keeping all these* [newsletters] *and I'll know what* [two-year-old] *Jessie needs when she goes to school.*
> (Rachael's mother)

Parents were conscious of the limited time they had to spend with their children. Some spoke of the need to give their child a break from schoolwork when they came home, or described the constraints on their own time. Yet, when it was suggested that the newsletters perhaps asked parents to do too much or "increased guilt", all but one disagreed strongly. It appears that the tone of the newsletter had minimized any feelings of compulsion. In contrast to the school situation, where teachers reported difficulties individualising work or giving examples in context for the children; parents' anecdotes were almost always in a natural context. The most common parent reaction to the newsletter was to include "mathematics talk" during their normal routines, without devoting extra time to mathematics. They were less likely to complete a suggested activity similar to "classroom mathematics", although some did and reported that they had enjoyed them.

> Father: *Routine things are easy because it is just remembering to say it, but occasionally we'd do homework and it would be part of that. But you've got to actually sit down ... and it is the time ...*
> Int: *Would you have been doing that anyway?*
> Mother: *Now we talked about* [mathematics]. *It made you more aware of what you were doing. Normally you just have the shape in front of you but you wouldn't particularly notice anything about it.*
> (Linda's parents)

Teachers' response to the newsletters

Teachers were also in favour of the newsletters but were unsure as to how universal the benefits were. For instance Wilma, (a Milland teacher) was enthusiastic about the benefits of the newsletter for a particular child with special needs, whose mother was given the newsletter before the class had tried the activity. "[The child] *is working at about an average level in the class. She wouldn't be there if it wasn't for her mother going through the work with her first."* But Wilma was more cautious about attributing the other improvements she had observed to the newsletters. For instance she noted that her whole class had been much faster in developing an understanding of one-to-one correspondence than in previous years, but she wondered if this was due to the newsletter or changes at the local early childhood facility.

Hannah felt that the newsletters were being read and used by her "good" parents but ignored by the others. She saw this as being of little value since she expected that the "good" parents were doing mathematics with their children anyway.

> *You can always pick the ones that get it from parents because they're the ones that are way ahead. My* [BSM cycle 8, module 2] *ones. I know the parents have put a lot into them before they even come to school and continue to do so. And that's why they are way up on 8-2.*

The key benefit of the newsletter identified by all the interviewed teachers was that the parents were shown the "correct" algorithms for the classroom mathematics. Teachers were not in favour of parents "teaching" and, when asked to comment, they discouraged parents from attempting. Yet they accepted that some parents would teach mathematics to their children no matter what advice was given. Teachers anticipated that the newsletter would minimize the problems caused, because the methods demonstrated by parents would be the same as those learnt in school.

Table 1: Differing perceptions of Junior School Mathematics

Teachers' Views	**Parents' Views**
Early mathematics involves a range of generic skills; comparison and relationships, shape, movement and position, classification order and pattern.	Early mathematics is about learning to count, learning to add and subtract, and later on; learning the times tables, before mastering algorithms for arithmetic with large numbers.
Delaying written mathematics gives children the opportunity to develop their mathematical ideas through guided exploration with materials.	Written mathematics should be introduced as soon as possible. Worksheet-type activities where children write numerals to represent the number of objects shown are the starting point for mathematics.
Children build their own knowledge of mathematics through experience. Teachers can assist through questions that prompt the child to re-examine their own understanding.	Children learn mathematics by being taught methods and by memorising basic facts. Parents can help by giving children extensive exposure to arithmetic facts at an early age.
Teaching mathematics in context is a noble ideal but very difficult to do in practice in the classroom. Planning is too extreme and real individualisation is impossible.	It is sometimes difficult to find time to complete set activities or mathematics homework but it is very easy to incorporate mathematical vocabulary and thinking into everyday situations.

Discussion

The interviews revealed a number of discrepancies between teachers' and parents' views of mathematics. These are summarised in Table 1. In many cases these differences are masked by the way in which both groups appear to view mathematics as "the forgotten subject"; of secondary importance to the development of language skills.

The regular newsletter proved to effective in modifying parental views of the first three items in the table, by giving parents exposure to the teacher's view of mathematics. Data from the interviews showed that newsletters expanded parents' view of mathematics (more than just arithmetic). The newsletters gave parents access to the teachers' justification for a delayed introduction of symbols and written recording of equations. Newsletters also encouraged parents to adopt the accepted "teacher view" of assisting children to develop mathematical understanding through questioning and prompting rather than rote learning basic facts. While parents still placed a great deal of emphasis on basic facts, in some cases this was modified by a focus on understanding.

In terms of the child's learning, the most important effect of the newsletter was the way parents were able to support classroom mathematics with more contextual problem solving. In contrast to the teachers, parents reported no difficulty in teaching *in a meaningful context.* Sharing the current classroom mathematics with home allowed parents to better support their child's mathematics *in a way teachers could not.* Parents showed skill, creativity and sensitivity to their child's needs. They reported that that it was "easy" to find time for contextualized mathematics in their normal routines *if they knew what the children were studying and how to apply it.* The findings of this research raises questions about the effectiveness of traditional mathematics homework tasks which often imitate the decontextualized mathematics or artificial exercises that are commonly seen in the classroom.

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Challenging beliefs about mathematics learning and teaching using an electronic learning community

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Prospective primary school teachers often see mathematics as a rigid, inaccessible subject. Their fixed ideas about the nature of mathematics can often impede their learning, and future teaching. This paper investigates a web-based intervention which encouraged dialogue about mathematics between an international community of mathematics educators and the prospective teachers' local learning community. Data from two surveys and from the prospective teachers' reflective journals show that this intervention encouraged prospective teachers to examine and evaluate their own beliefs.

Introduction

Prospective teachers of mathematics in the primary school often have negative experiences in their own learning of mathematics (Pateman, 1989). Further, many have ideas about mathematics education that are at odds with the views espoused by mathematics educators interested in reforming the way mathematics is taught (Australian Education Council, 1991; NCTM, 1991). The literature in mathematics education shows that many primary school teachers and prospective primary school teachers see mathematics as a fixed and sequential body of knowledge which is most effectively learnt by rote, algorithmic and repetitive procedures (Mayers, 1994). Burton (1996) suggests that the teaching of 'immutable mathematics' has led to widespread marginalisation and failure for its learners. She propounds the view that the very socio-cultural nature of mathematics has led to a differentiation amongst its learners of those who can engage with the traditional presentation of the subject and those who are unable to do so. Her views are shared by many mathematics educators (Dengate and Lerman, 1995; Ernest, 1991) and a reform movement in mathematics education has evolved with aims of making mathematics accessible to more people.

It has been suggested (Crawford and Deer, 1993) that prospective primary school teachers' views of mathematics as instrumental might be modified "if opportunities for active and reflective participation in the learning process are provided" (p. 30) in the teacher education program. Intervention studies conducted in teacher education programs show some success (Schuck, 1997; Lappan and Even, 1989). However, many teachers still appear to start their teaching careers with the views that they held when they started their teacher education preparation. The implications of teaching with the instrumentalist views described above are that primary school students are introduced to a view of mathematics which accepts that access to the power of mathematics is available to only a few 'elite' students and denies others an entry into a world of mathematics that is culturally and socially relevant for them. If we wish more students to have access to the power and beauty of mathematics, a different vision of mathematics needs to be presented to them from primary school days onwards. For this to happen, primary school teachers, and prospective primary school teachers, need to challenge their own views of mathematics so that they become open to visions of mathematics as a dynamic, creative and problem solving discipline in which procedural methods are merely means to ends, rather than the ends themselves.

This paper considers an intervention which challenged prospective primary school teachers' beliefs about the nature of mathematics and mathematics learning through the medium of a web-based computer conferencing tool. Details of the intervention are described and the results are discussed. In addition, a few factors that inhibit change in beliefs are identified and discussed, in particular, the influence of the practicum on prospective teachers' beliefs.

The Context of the Study

Prospective teachers in an initial teacher education program for primary teachers at the University of Technology, Sydney (UTS), have a sequence of four semesters of mathematics education which begins with Mathematics Education 1 in the second semester of their first year. In 1997, one hundred and eighty five first year prospective teachers in the Bachelor of Education program at UTS enrolled for Mathematics Education 1. A major component of this subject was a substantial intervention aimed at challenging prospective teachers' beliefs about the nature of mathematics. The intervention was ongoing through the semester.

The study investigated whether the intervention, using web-based conferencing and an international community of learners, would encourage prospective teachers to examine and challenge their beliefs about mathematics, and mathematics teaching and learning.

The Intervention

The characteristics of the intervention were that it be collaborative, that prospective teachers be given access to a large number of views about the nature of mathematics, and that the material be current. relevant and challenging of prospective teachers' beliefs about mathematics.

In the first lecture prospective teachers were given two focus statements, which formed the basis of the computer conference. These were:

On the nature of mathematics:

> "I'd describe maths as the calculation of certain things to do with numbers, and the use of certain formulas and methods, simplifying, counting and subtracting and things like that." (Maria, first year prospective primary schoolteacher)

On the cultural context of mathematics:

> "Mathematics is universal, objective and unchanging. It is independent of social, cultural and political values."

The first statement had been made by a first year student teacher in a previous research study done by one of the researchers, and the second one was a compilation of comments made in that same study by a number of prospective teachers (Schuck, 1996).

The prospective teachers were then given the task of forming groups with five or six members and choosing one of the two focus statements for their study and response. They were introduced to the computer conferencing tool and the requirements of the task. Three types of material were made available to the prospective teachers: a selection of readings to be held in the library collection; a number of websites with relevant material; and responses to the focus statements by a number of respected mathematics educators, which were placed at the discussion site of the conferencing tool. The last set of materials was written specifically in response to the focus statements, by mathematics educators from a number of countries, including Fiji, New Zealand, the United Kingdom and Australia. These educators had been invited to respond to the statements by the researchers/subject developers.

Prospective teachers were required to read from each of the three sets of material, and then work collaboratively to formulate a group response to the chosen statement, based on their understandings of the material, and their ensuing discussion as a group. They were then required to post their response into the discussion site of the conferencing tool on the web. Time was allowed in the subject for group meetings and collaboration. After the groups had all posted their initial responses to their chosen statement onto the conferencing tool (approximately halfway through the semester), they were required to read each others' responses to this statement, displayed in the discussion site, and then, in the same groups as before, formulate a second response to the statement that took other groups' points of view into account.

Towards the end of the semester, the entire first year cohort undertook a three week practicum and was required to teach a number of mathematics lessons during that time. Finally, in the second last week of semester, after prospective teachers had posted their second response on the conferencing site, class discussions were held about the various responses to the given focus statements and the implications of these responses for teaching.

Methodology of the Study

Participants

One hundred and eighty five prospective teachers initially participated in the study. One hundred and sixty five of them responded to a survey, which asked for some biographical and demographic data. Of the prospective teachers responding to this survey, 101 were eighteen to twenty years of age, the remaining participants being in a range from twenty one to forty seven years of age. One hundred and forty prospective teachers were female, twenty five were male. The two researchers were the academics responsible for the subject at that time.

Data collection methods

Prospective teachers were surveyed in their first lecture about their beliefs about mathematics and mathematics education. The survey was anonymous, with prospective teachers being asked to supply code names so that each individual's survey results could be matched with the results of a later survey. Prospective teachers were then given the two focus statements that formed the basis of the web mediated discussion and asked to write down their first thoughts in response to these statements. As prospective teachers were required to maintain a reflective journal throughout the semester, the responses to these statements were in the nature of journal entries. These were not submitted to the researchers at that time. The intervention was then carried out over the semester and prospective teachers were requested to keep journal entries on the group process and their thoughts about mathematics. At the end of the semester the journals were collected by the lecturers and permission gained to use them as data. Together with the journals, prospective teachers were asked to submit summaries of the journal notes, documenting any critical incidents or new learning that had occurred during the semester.

In the second last week of the semester, the prospective teachers were given another survey which included the same set of questions about beliefs as had been asked at the beginning of the semester. The means and standard deviations for the questions on beliefs on the two surveys were then calculated. Prospective teachers were asked to use the code name chosen at the beginning of the semester.

Analysis

Four categories were created by the researchers to collect information from the reflective journals about areas of interest. The first category was one called *Beliefs*. It held material about the students' beliefs about mathematics and mathematics teaching and learning. The other categories were: *Technology* (material discussing the value of the conferencing tool, or any issues relating to this technology), *Evaluation* (material evaluating any aspect of the subject), and *Workshops* (material relating to another component of the subject). See Foley and Schuck, 1998, for discussion of the results from the *Technology* category. The present paper focuses on the first category, *Beliefs*.

A few reflective journals which expressed a view that appeared representative of a majority view were selected to be copied, and any journals that expressed a view that was substantially different in any way from other views were also copied. Two journals were read by both researchers to ensure that they were being categorised in the same way. The researchers then categorised all the selected journals. The contents that had been marked as *Beliefs* were used to supply data for this paper.

The two sets of surveys were also analysed for the information they supplied about beliefs. Prospective teachers were required to score each statement on a Likert scale. Examples of beliefs statements were the following two statements: "mathematics does not change with time, only the ways of teaching it change"; and "mathematics is value free and objective". A mean and standard deviation for the group, on each statement about beliefs, were calculated. A t-test for paired samples for means was used on the group of 18 belief statements, to test for statistical trends. Each item was also tested for significance using a t-test on the responses on each survey for that item. As only fifty one prospective teachers had put their code names on both surveys, all statistics were calculated for that subgroup ($n=51$) of the whole cohort, to ensure that the groups were paired.

Results

For each individual item about beliefs, the change in mean values on the two surveys is small, and only three of the 18 changes are statistically significant with $p < 0.05$. However, when the entire group of beliefs statements are considered, there is a clear trend in the direction of change of beliefs towards a more liberating view of maths ($t=3.7$, $p<0.01$).

Prospective teachers' journal entries offer more in-depth information about the intervention and other factors influencing prospective teachers' beliefs. Many prospective teachers commented in their reflective summary, written at the end of semester, on how limited their thinking had been about mathematics when they initially responded to the two statements. For example, the following text from one student's journal and summary show this kind of development. Jenny's (pseudonym) initial response to the first focus statement was as follows;

> I agree with Maria's statement. My views on maths are similar. My thoughts on maths are basically just different ways of applying figures and numbers together in order to get an answer. I think this is due to the way I was taught in maths, from a young age, through to a teenager, I found maths to be a very passive learning experience, I was told what to do and how to do it and that was the end of it. I kept on thinking this is not relevant to life, unless you wanted to become a mathematician which I had no intentions of becoming one [sic]. I don't understand or appreciate maths, therefore I can't see past the numbers, formulas, etc. Mathematics makes very little sense to me especially at the high school level - years 11 and 12.

At this point Jenny is showing the sort of thinking about mathematics that is limiting for her, and will prevent her from sharing a more dynamic view of the nature of mathematics with her primary school pupils when she starts teaching.

The next extract of text is taken from Jenny's journal after she has started reading the material provided to challenge beliefs:

> After the completion of this article [NCTM, 1989], I discovered mathematics means so much more than just numbers. It showed me the importance of the process - discovery/journey of mathematics than to get the answers. It can also link to different KLAs [Key Learning Areas], while, prior, to reading this article, I thought mathematics was an isolated subject on its own. Not forgetting maths should have relevance to life, therefore it is easier to understand. This article has given me a whole new approach of looking [at] and thinking [about] mathematics.

Finally, Jenny wrote in her summary about the new learning occurring during the semester:

> At the start of semester two I had a very narrow view of mathematics, thinking it was just boring numbers and formulas. My view was very similar to those [sic] of Maria's quote. However, as I progressed through this subject I found my views changing at a moderate pace. Ö
>
> The second group discussion gave me the opportunity to read other groups' responses which gave me an even broader knowledge on the meaning of mathematics.
>
> In conclusion, my views have completely changed. They are no longer limited. ....

So for Jenny, both the material supplied by the lecturers and the responses of her peers helped her to challenge her own beliefs. Another student also wrote enthusiastically about the value of the group discussion that was a central feature of the intervention. She wrote in her summary:

> To be honest I think that the maths discussion tested my beliefs and ideas about maths more than any test could.
>
> Thank you who ever reads this journal because I would never have thought of some of the points that came up through other people's readings and ideas, and also they really began to challenge my own.

And finally, a male student, who had shown some impatience with the way the subject had started, as shown by his comments, written in his journal about the first class held for the subject:

> Right now I'm hoping that my attitude improves greatly because I'm thinking "When the h.. are we going to do mathematics?". Ö I will try to understand why we are talking about everything but maths. I like maths and I want to do some.

He seemed to show some awareness that he had developed in his thinking a few weeks later: "Have realised by reading for the assignment that I've already started to change my thinking about mathematics. It's a lot more complex than suggested in the quote [Maria's statement]Ö."

And in his summary:

> The mathematics readings, or one in particular, opened my eyes to something else. I didn't realise that different cultural backgrounds made a difference to people's concepts of mathematics. Obviously people living in a third world country with little access to education will not be as good at it as others who have access. But I thought this was only an educational deficit, but then to read about how playing particular games in

> India, girls become better at mathematics [an article from the web: http://ms.mathscience.k12.va.us/lessons/kolam.html], or hearing in the tutorials that Fijian children have little concept of mass [maths?] and number because they give and share everything [sic].[This last comment was sparked by a response by a Fijian mathematics educator to the statements on the conferencing tool.]
>
> I think it is a good idea to keep a diary, it makes one think about what they have done during the day and reading back over it I can see where my ideas have changed about the subject. For example I can remember telling you that I basically agreed with Maria's quote, but I thought it could have been written a little better. By the time I had finished the first reading I realised that Maria and I were totally wrong. If not wrong, narrow minded in our views of what mathematics is.

There are many rich examples in the text of how prospective teachers felt that their original thinking about mathematics, as evidenced by their responses to the quotes in the first lecture, were very limited, and how they felt they had changed their thinking to some extent as a result of the discussion and sharing of material in their mathematics education community. There were also a few examples showing that even while beliefs may have been challenged by the intervention, the experience on the practicum often led to some tension between the approaches used in the university subject and those used by the supervising teacher. The following extract shows this sort of thinking:

Jake (pseudonym) started his journal by writing about his goals for the subject:

> There are many goals I would like to reach by the end of Mathematics Education 1. ...
>
> It should be emphasised through lessons that the answer is NOT the most important aspect of maths, but the method from which it derived - this method or technique I wish to begin to understand, so I can teach mathematics successfully.

Later in the journal, Jake responded to an aspect of his group's discussion:

> Our conversation confirmed that attitudes are constructed through either positive or negative experiences in mathematics, the fact that, unfortunately, some teachers seek only correct responses.

But his notes on the practicum in his summary seemed to be expressing an opposite view to the above, and some dissatisfaction with the workshop approach used in Mathematics Education 1:

> The greatest learning experience in the field of mathematics occurred during practice teaching. The class although challenging, allowed me to experience what it's really like to teach... Initially I made several errors, I took too long in marking the answers (should have been informed on how to mark work for optimum results) and not knowing how to answer certain questions, which I had to figure out myself.
>
> These failures set up successful maths lessons which were highly controlled in terms of explanation, modelling and exercises. The methods I used were from my assisting teacher, as she recommended that I mark the work fast, seek exercises that are challenging and to explain concepts thoroughly. These

may not sound difficult, but we never discussed this in mathematics.

Discussion and Conclusions

It would be ambitious to expect that beliefs held for many years could be changed in the space of one semester. However, there was sufficient indication in the data from the surveys and the journals to show that the intervention was successful in encouraging prospective teachers to become aware of their own beliefs, and that the process had begun to challenge their beliefs. Prospective teachers showed that they had increased awareness of the influence of school students' cultural backgrounds on their learning of mathematics, the differing nature of mathematics in different cultures, the importance of process as well as solutions, and the possibilities of integration across the curriculum.

It should be noted that the web-based conferencing tool was just that - a tool or device which was used as the medium for the intervention. In itself, it was not instrumental in changing beliefs but provided a vehicle through which contact with the international community of mathematics educators, and with the local community of learners, could occur with immediate feedback. The community of mathematics educators participated in the discussion in much the same way as the prospective teachers - they responded to the focus statements, just as the prospective teachers did, and they had the opportunity to participate in the discussion throughout the semester. Another advantage of the conferencing tool was the ease with which prospective teachers could read each others' responses to the focus statements. Prospective teachers commented that the knowledge that their response was going to be read by others encouraged them to think more carefully about the material they would submit. Having the responses placed on the web also meant that prospective teachers could use each other as the learning community and benefit from the careful thoughts of other groups. It encouraged discussion in the whole learning community of the cohort, as opposed to dialogue between the group and the lecturer. Hence the use of the conferencing tool made access to material easier and more immediate. For a detailed discussion of the value of the conferencing tool, see Foley and Schuck (1998).

Finally, some comments from the student journals clearly show a tension between approaches used in the university based part of the subject, and the experiences occurring during the practicum. Where supervising teachers saw mathematics differently from university teachers, some conflict was evident in the beliefs of the student teacher over the two periods. The implications for teacher educators wishing to challenge beliefs, are that communication between university staff and school staff must be improved, so that teachers are aware of the content and approaches of the university subjects, and academics are aware of the approaches, beliefs and constraints operating in the school classroom. With this knowledge, a better understanding of the role of the practicum can be negotiated.

In general, then, the intervention was successful in challenging student teachers' beliefs, but more attention needs to be paid to the influence of the practicum on the learning occurring in the mathematics education class at university.

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Mathematics textbooks: Messages to students and teachers

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The set textbook has a strong influence on what takes place in the mathematics classroom, especially at the secondary level. This paper reports on the preliminary stage of a project aimed at developing more effective textbook presentations. In this phase of the project, textbooks are being examined to identify the types of messages about mathematics and its teaching and learning inherent in their presentations and to identify the sources of the underlying messages.

Introduction

Textbooks are widely used in mathematics classrooms in Australia and many other parts of the world at all levels of schooling. Fauvel (1991) described the textbook as one part of a "book / pupil / teacher triangle" (p. 111) and noted that the approach in the textbook is a constraint on what the teacher can do in the classroom. In a survey of textbook usage across a range of subject areas in Australia, Morris (1989) observed that it was often the teacher who read and learnt from the textbook and not the students. Shield (1991) reported on a study of seven mathematics teachers and their uses of the set textbook with their year 8 classes. In this study it was the teachers and not the students who read the textbooks and used them in their pedagogic decision making. Students use was mostly restricted to doing the extensive sets of exercises which accompanied each section of the book, with occasional reference to a worked example. It was also noted that the way the textbooks were written did not provide encouragement for students to seek information.

There have been many calls for an increase in the range of strategies used in the teaching and learning of mathematics. Examples include the documents *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM], 1989), *Everybody Counts* (National Research Council, 1989) and *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991). These reports generally call for a broadening of teaching approaches, a greater emphasis on understanding, and a stress on the usefulness of mathematics through applications and problem solving. However, the teaching of mathematics in schools has generally been at variance with the ideas being developed in these documents. For example, Baroody and Ginsburg (1990) describe the "tell-show-do" approach in which the teacher firstly tells the students what they need to know, follows this by showing an example, then the students do a number of similar examples themselves.

The study reported in the present paper is the preliminary phase of a project to develop more effective models for the presentation of print and eventually on-line mathematical resources that are more closely aligned with recent reports and curriculum documents and with research into the ways students learn mathematics with understanding. Specifically, this part of the study looked at the presentations in some current school mathematics textbooks at the lower secondary level with a view to identifying the inherent messages about mathematics and its teaching and learning and the sources of these messages. The discrepancies with the thrust of current curriculum documents was also examined. While it is yet to be determined just how textbook mathematics can be presented in order to take full account of the current directions of mathematics education, it is important that teachers using traditional textbooks are aware of the hidden messages contained in them so that they may address such issues in their teaching.

Background

The analysis of the textbooks presentations is based on a number of ideas about mathematics and its teaching and learning which are briefly discussed here. Skemp (1976) introduced the terms "relational understanding" and "instrumental understanding" in a discussion of mathematics learning. Relational understanding describes the state of knowing in which the learner builds up an interconnected, structured knowledge. Instrumental understanding describes the type of knowledge in which the learner acquires a set of fixed procedures and rules which exist more or less as individual elements of knowledge to be applied to very specific situations. An alternative formulation of this idea was provided by Hiebert and Lefevre (1986) in their discussion of "conceptual" and "procedural" knowledge in mathematics. They defined conceptual knowledge in similar terms to those of Skemp, that is, as a "connected web of knowledge" (p. 3) and requiring the learner to recognise "its relationship to other pieces of information" (p. 4). Hiebert and Lefevre described procedural knowledge in terms of two kinds of information: knowledge of the language and symbols and the conventions for their use; and knowledge of the rules and procedures used for solving mathematical problems.

Bell (1993) describes "mathematical activity" in terms of a cycle of "mathematization, manipulation, and interpretation" (p. 6). The three stages involve the recognition of a mathematical relationship and its symbolic expression, the manipulation of the symbols to uncover some new aspect, and the interpretation of the result in terms of the situation. Bell noted that traditional mathematics teaching spends most time on the manipulation stage of the cycle.

Teachers' instructional practices are closely related to their views of mathematics and its teaching and learning. Thompson (1992) found support for this relationship, but also showed that the relationships among a teacher's views, beliefs and instructional preferences were highly complex. A teacher's view of mathematics and its teaching and learning may be described as lying somewhere along a continuum. At one extreme, the view of mathematics has been variously described as formalist (van Dormolen, 1985) or instrumentalist (Skemp, 1976; Ernest, 1989). Those holding this view of mathematics see it as a system of facts, rules, theorems and skills in which intuition plays no real part. The associated view of teaching and learning is of a transmission process or a conveyance-container model (Mason, 1989). This model characterises mathematical knowledge and skills as existing entities which have to be acquired by the learner. The emphasis is on the learning of technical terms and their definitions, skills and techniques with much symbolic recording and manipulation.

At the opposite end of the continuum is the view of mathematics as an open, dynamic field which is constantly changing and expanding, an "activist" view (van Dormolen, 1985). Doing mathematics is a process of exploration using activities such as classifying, formalising and quantifying. Intuition is an important element. Teaching entails facilitating the use of these processes by the students. It is possible for an individual teacher to adopt a view of school mathematics and its teaching somewhere between the two ends of the continuum. Cobb (1988) described a continuum of teaching approaches from negotiation (active construction) to imposition (transmission) and noted that most mathematics teaching tended towards the imposition pole.

After more than six years of formal education, secondary school students have formed their own conceptions of what it means to study mathematics in school. This will have a considerable influence on their acceptance of the experiences provided by the teacher. Cobb (1986) provided the example of "students who have constructed instrumental beliefs about mathematics (Skemp, 1976) anticipate that future mathematical experiences will 'fit' these beliefs" (p. 4). A teacher attempting to present mathematics in a more "activist" way may experience resistance from such students. Garofalo (1989) discussed the common beliefs about learning mathematics held by secondary school students in the United States. These include:

> Almost all mathematics problems can be solved by the direct application of the facts, rules, formulas, and procedures shown by the teacher or given in the textbook. Only the mathematics to be tested is important and worth knowing.
>
> Mathematics is created only by very prodigious and creative people; other people just try to learn what is handed down. (Garofalo, 1989, pp.502-503)

In a study of Australian students, Herrington (1990) found similar beliefs and stated that "it is apparent that many students see practising the same question over and over again, or copying notes from the blackboard, as the best way to learn mathematics" (p. 15).

Mathematics textbook analysis

A number of authors have reported on methods of examining various aspects of mathematics textbooks. Fauvel (1991) analysed three early textbooks (from the years 1543, 1743 and 1910) from the viewpoint of what he called the *tone* of the texts, that is, "how the author has worked to influence the way that readers must respond to the book to benefit in the way intended" (p. 111). He described textbook writers as "teachers at a distance" (p. 116) and examined the pedagogic techniques used in the texts. Morgan (1996) employed a linguistic approach to mathematical texts. While her work has focussed mainly on texts written by students, the approach is equally applicable to published textbook material. Using the work of Halliday (1973) she described three aspects of text, namely: the *ideational* function which addresses the nature of mathematics and mathematical activity including the role of people in its creation; the *interpersonal* function which considers the roles and relationships of the reader and author; and the *textual* function which considers the way the text is constructed. Morgan's discussion elaborated the difficulties students appeared to find in producing written mathematical texts in extended assessment tasks to match the expectations of their teachers.

Van Dormolen (1985) provided a method of analysing the content in mathematics textbooks. Terminology and categories used by van Dormolen were adopted and extended in a coding scheme developed by this author in earlier studies (e.g., Shield, 1996) with some minor renaming of categories. A brief overview of the coding scheme textual categories and descriptors is provided. These provide a language for the discussion of the characteristics of mathematical texts.

Elaborations: *kernel* **(K)** - general expressions that have to be learnt - rules, definitions and procedures; *exemplar* **[E]** - a worked example of the procedure or an example of the concept, which is presented with at least one of the following components: *symbolic representation* **[S]** - numerical and/or algebraic symbols: *verbal description* **[V]** - the specific worked example given in words: *diagram* **[D]** - a line drawing to support the exemplar; *table of data* **[T]** - a collection of data of some kind arranged systematically: *convention* **[C]** - a verbal statement of a generally accepted practice; *graph* **[G]** - a Cartesian line graph, bar graph, histogram, etc; *goal statement* **[G]** - an identification of the concept or procedure that is the subject of the text, often a heading; *justification* **[J]** - an attempt to show something to be correct by referring to known principles; *link to prior knowledge* **[L]** - a description of a prior mathematical skill needed for the new procedure, or a description of an everyday example of the concept or procedure; *practice exercises* **[P]** - a set of questions for the reader, modelled on the exemplar.

All verbal and symbolic statements in the text, including kernels, are described in terms of their "aspect of mathematics" and "level of language" as described below.

Aspects of mathematics: *theoretical* **[T]** - theorems, definitions, axioms - parts of the structure of the topic; *algorithmic* **[A]** - explicit methods or "how to do" a specific operation or procedure; *logical* **[L]** - statements about the way one should work using the theory - an extension of the theory; *methodological* **[M]** - "how to do" rules but of a more general, heuristic nature than algorithmic - the rule needs interpretation to provide an answer; *conventional* **[C]**- conventions, how to name a diagram, write a proof.

Levels of language: *particular* - language used in discussing a specific example or case. *generalised* - language conveying the general meaning of a concept or procedure and not related to specific examples. Within each level of language, the statements may be either *procedural* or *descriptive*: *procedural* - step by step instructions; *descriptive* - stating the meaning of an idea or the appearance of an object or diagram. There are therefore four language descriptions: *particular procedural* **[PP]**; *particular descriptive* **[PD]**; *generalised procedural* **[GP]**; *generalised descriptive*.

A text analysed

To illustrate the analysis being adopted in the present study, a section of text from a published textbook by this author (Shield & Wallace, 1988) has been used. The chosen textbook has many similarities to other published mathematics textbooks which are being analysed in the project. While the intention is not to be critical of the writers of successful (in terms of sales) textbooks, it is inevitable that criticisms will be inferred from the findings of the analysis. However the constructive intent of the criticisms is discussed in the conclusion of this paper. Two pages from the textbook are reproduced in Appendix 1 with codes for elaborations (underlined), aspect of mathematics (boxed) and level of language (circled). The two pages were chosen as typical of pages devoted to the exposition of new material. The pages immediately preceding and following those illustrated are discussed later.

While this text contains a little more verbal material than some others, it is still rather "dense" in terms of the number of ideas that have to be remembered by the reader. On the two pages in the appendix there are 4 statements classified as *kernels*, being *generalised* statements of definitions and procedures, for example: "A common factor of two numbers is a factor which occurs in both numbers." The aspects of mathematics being developed are *theoretical* and *methodological* in the kernels, followed by the *algorithmic* use of the mathematics in the worked examples. Each kernel is illustrated with at least one specific example (*exemplar*). There is no use of *justification* in the presentation and unusual cases are not considered. The worked examples of the two main skills being developed are followed by sets of practice exercises. The exercises can be completed by the reader by using the worked examples as models.

The approach to the skills is purely instrumental and at this stage there is no indication of possible uses for the skills being developed. On the previous pages in the textbook, the reader is reminded of what should be prior knowledge of factors, products and prime and composite numbers. There is an investigation using the ideas of the "sieve of Eratosthenes" which talks about Eratosthenes being a Greek mathematician of the third century BC and a friend of Archimedes. The remainder of the chapter after the pages reproduced in Appendix 1 is devoted to division, rules for divisibility, and approximate calculations. There is no further reference to highest common factor.

Interpretation of the text

The main aim of the preliminary phase of the project was to identify the underlying messages which the textbooks convey to teachers and students. The above type of analysis provided the basis for reflection on the views of mathematics and its teaching and learning inherent in the text. It was also necessary to consider the context in which such a text would be used to inform further analysis. Generally the set textbook is the only resource in the students' possession and, as mentioned earlier, is fundamental in the planning of many teachers. While the context, in terms of the views of the teacher and the range of experienced offered in the classroom, may vary considerably, the mathematics textbooks being analysed appear to have been written with a "tell-show-do" approach to mathematics in the minds of the authors. The repeated use of kernels, worked examples and exercises in the example text analysed here, fits this pattern.

The use of kernels expressing theoretical and methodological aspects of mathematics in generalised descriptive and procedural language conveys the idea that mathematics consists of complete and established principles which have to be acquired by the learner. There is no "mathematical activity" (Bell, 1993, p. 6) with its cycle of "mathematization, manipulation and interpretation" evident here. The ideas are presented in complete form and illustrated abstractly. While there has been some attempt to develop the ideas from prior knowledge of factors, the end result is an "instrumentalist/formalist" view of mathematics presented in a transmission mode. The lack of any reason for doing the procedure and any real use of the procedure reinforces this view. The presentation shows little evidence of the development of conceptual knowledge and concentrates on the two kinds of procedural knowledge identified by Hiebert and Lefevre (1986), that is knowledge of the appropriate language and symbols and knowledge of the rules and procedures as demonstrated in the kernels and exemplars. If this presentation were to be followed closely by the teacher, there would be little space for the active construction of the ideas by the learners.

The sample text may be characterised using Halliday's (1973) aspects of text. The *ideational* function addresses the nature of mathematics and mathematical activity which is essentially instrumentalist. It also includes the role of people in the construction of the mathematics. While the sample textbook and all the others so far analysed include references to people in brief historical snippets, there is no real feeling that the mathematics being learnt is a human endeavour. Nominalised mathematical objects are manipulated logically. The *interpersonal* function considers the roles and relationships of the reader and author. The sample textbook presentation has an authoritative tenor in which the authors are transmitting the necessary knowledge to the reader with the reader expected to just accept what is being presented. While the sample presentation, and most others analysed, include statements of the type "we can see that . . . ", the use of the inclusive "we" does not really have the function of including the reader in the development of the idea. There is little else that the reader can do but accept what is being presented and practice the procedures using the exercises provided. It would be interesting to know how mathematics learners interpret such inclusive statements, or if they have any impact at all. Finally, the *textual*function considers how the text is constructed. As discussed earlier, the sample text consists mainly of sequences of kernels, exemplars and practice exercises typical of a tell-show-do approach. Generally this approach is likely to match the expectations of the teachers and students using the text.

Conclusion

In this paper, a method of analysing the content of mathematics textbooks has been outlined. The mathematical teaching function of each statement in a text can be defined as a kernel, exemplar, link or other categories. Identification of the aspect of mathematics being expressed and the level of language being used provides a characterisation of each mathematical statement. From this analysis, the underlying messages about the nature of mathematics and its teaching and learning that the text might convey to teachers and students may be inferred. While the influence of the textbook on mathematics classrooms is known to be strong, it is not possible to know the extent to which the underlying messages influence classroom proceedings in individual cases.

The textbooks analysed so far are generally similar to the sample text demonstrated in this paper. Clearly, the textbooks themselves do not convey the intent of recent reports and syllabuses, even though they were written in response to these documents. While it may not be possible, in a printed form, to totally meet the needs of new syllabuses, it should be possible to develop textbook presentations which come much closer than at present. It has been shown that many mathematics teachers at all levels rely heavily on the textbooks for their curricular decisions and while some may be able to recognise and overcome the limitations of the textbooks, many cannot. There is also the question of the role of the learners in using the textbooks and their expectations about what doing mathematics in school entails. The project aims to further such questions.

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Factorisation

T GP

K To factorise a number means to write it as a product of two or more factors, e.g. 12 = 4 × 3, 24 = 2 × 3 × 4. PP ES

EV Any composite number can be factorised. A prime number such as 23 can be written as 1 × 23, but in mathematics this serves little purpose. Usually, numbers are written as a product of all prime factors, e.g. T PD C

ES
36 = 3 × 12 (3 prime, 12 composite)
= 3 × 3 × 4 (3 prime, 4 composite) A
= 3 × 3 × 2 × 2 (all prime)

EC A factor tree is a way of organising this. C GD

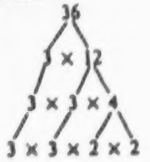
Example

Write (a) 36 and (b) 84 as products of prime factors.

Solutions

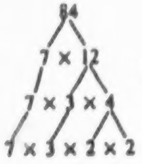
ES

(a)



36 = 3 × 3 × 2 × 2

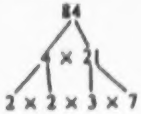
(b)



84 = 7 × 3 × 2 × 2

A

EC Note: It does not matter which factors you start with, e.g. for (b) above: A GD



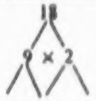
84 = 2 × 2 × 3 × 7

SHARPEN YOUR SKILLS 3

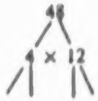
P

A. Complete these factor trees.

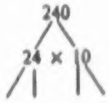
1.



2.



3.



A

B. Write each of these numbers as a product of prime factors using a factor tree.

1. 30
2. 42
3. 27
4. 210
5. 240
6. 98
7. 216
8. 315
9. 960
10. 968

174

HIGHEST COMMON FACTOR

T GD

K A common factor of two numbers is a factor which occurs in both numbers. (For this definition, name the: *item*, *class*, and *feature*.)

For example,

ES
24 = 2 × 2 × ∠ × 3 (prime factors)
36 = 2 × 2 × 3 × 3

EY From this we can see that there are some common factors of 24 and 36—two 2s and one 3.

K The highest common factor (HCF) of two numbers is the biggest number which is a factor of both numbers. (Definition: *item? class? feature?*) T GD

K The HCF can be found by finding all the prime factors which are common to both numbers. M GP

Example

Find the highest common factor (HCF) of (a) 24 and 36 and (b) 48 and 18.

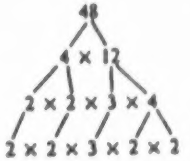
Solutions

(a)
24 = 2 × 2 × 2 × 3
36 = 2 × 2 × 3 × 3

ES The common factors are 2, 2 and 3.

HCF = 2 × 2 × 3
= 12

(b)
48 = 2 × 2 × 2 × 2 × 3
18 = 2 × 3 × 3
HCF = 2 × 3
= 6



A

EC Note: It is best to write the prime factorisations of the two numbers with the prime factors in increasing order of size. M

SHARPEN YOUR SKILLS 4

P

Find the highest common factor (HCF) for each of these pairs of numbers.

1. 8 and 20
2. 24 and 40
3. 42 and 14
4. 90 and 60
5. 24 and 48
6. 36 and 16
7. 78 and 26
8. 50 and 20
9. 108 and 72
10. 27 and 63
11. 15 and 18
12. 24 and 40
13. 36 and 60
14. 45 and 75
15. 28 and 42

A

175

LOWEST COMMON MULTIPLE

The lowest common multiple (LCM) of two numbers is the smallest number which is a multiple of both numbers. (Definition: *item? class? feature?*)

BEST COPY AVAILABLE

Problem Solving Through Problem Posing: The Experience of Two Teacher Education Students

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Teacher education students in their first year of university were invited to take part in an eight week program involving solving problems and posing problems on the basis of the given problem. Data concerning two of the eight students who agreed to participate, have been reported in terms of their problem solving strategies, their progress in problem solving skills and their expressions of cognitive and affective factors related to the study. It was found that the requirement to pose a similar problem to a given problem helped clarify thinking and that other important factors were the realism of the problem, the capability for visualisation and the nature and speed of feedback.

Despite over a decade of interest and emphasis on problem solving as a strategy for teaching mathematics and as a life skill, or perhaps, because of it, problem solving in schools appears to either be viewed sceptically as too hard or in an unenlightened way as being fairly routine. Indeed, many teachers who were quite excited about the possibility of incorporating problem solving into their daily classes in a way that would grasp the imagination of the students, have lapsed back into the habit of treating problem solving in a routine way. Whichever situation occurs, it is disappointing to note that the approach which held so much promise, challenge and excitement has reverted to the chore it was always considered to be. Even teachers and students who have maintained their enthusiasm for the use of problem solving as an important skill, sometimes feel overwhelmed and lack the confidence to explore the art of problem solving creatively. This can be extremely frustrating to the student and cause loss of self-esteem in the teacher.

Problem solving as an essential element in school mathematics has been advocated since the early 1980s by such seminal documents as the *Agenda for Action* (National Council of Teachers of Mathematics, 1981), the Cockcroft Report (1982) and the *National Statement on Mathematics for Australian Schools* (1989). The best indication of the framework within which this emphasis was to be implemented appeared in the report of a national workshop in relation to the Australian Mathematics Education Project (1981) and has been used in several states in their curriculum documents and other professional development activities. This framework lists three different aspects of teaching problem solving in schools, viz. teaching for problem solving, teaching about problem solving and teaching through problem solving. Teaching for problem solving means ensuring the learner acquires the skills, understandings and knowledge necessary for the task of problem solving. Teaching about problem solving means ensuring the learner has at his or her disposal the strategies or heuristics necessary for problem solving. Teaching through problem solving means using problem solving as the methodology through which mathematics is taught. Resources and research have tended to focus on the first two of these aspects. The third has been considered quite difficult except perhaps in the early years of schooling. One example of a positive result in teaching through problem solving is given by Villasenor and Kepner (1993). They used two groups of first-grade teachers and their classes. One group of teachers taught their classes using a problem solving approach and the other was a control group. They found that the experimental group performed better in problem solving than did the control group. This result seems to indicate that teachers who make the effort and are appropriately supported can make a difference in their students' problem solving ability. The researcher considers this result needs to be extended to problem posing as a complement to problem solving and therefore a possible tool for achieving the use of problem solving as a methodology for teaching mathematics.

In considering the reasons for a regression to routine treatments of problem solving in schools, the researcher became aware of two factors. One was the solver's lack of success in the problem solving process and the other was the solver's lack of willingness to attempt new methods and strategies. It was conjectured that the latter difficulty might well be overcome by students becoming more adept and creative in the solution of problems through the medium of posing problems.

The obverse side of problem solving is problem posing, and again, there have been several (Skinner, 1990, Brown and Walter, 1990) in the past decade who would herald problem posing as a panacea for deficiencies in mathematical thinking. Sometimes this is related to the notion of mathematical investigations in which learners are asked to pose problems about a particular situation, which is either well structured or loosely structured or having no structure at all, if that is possible. Other times, learners are given no assistance at all and are asked to pose problems on a fairly random basis. There are several other variations of the use of problem posing as a means of developing mathematical thinking and therefore of enhancing problem solving. Silver, Mamona-Downs, Leung and Kenney (1996) studied the problem posing ability of middle and prospective secondary school teachers in connection with a complex situation before, during and after attempting to solve a problem within the same setting. They found the subjects generated a large number of varied problems, more before the solution process than during or after. Another study by Silver and Cai (1996) asked middle school students to write questions in relation to a story situation that could be answered from the information given, This also produced a large number of problems though not all solvable. This same study also looked at the linguistic and mathematical complexity of the problems posed.

Ellerton and Clarkson (1996) claim that research in this area suffers from a lack of a research framework and give an account of the types of research that have been carried out in the last fifty years on problem posing. Much of the emphasis seems to have been on the development of skills in problem posing in a scientific sense - looking for the questions one should be asking to carry out appropriate research into a problem (p. 1010). They do, however, stress the importance of problem posing as a skill. Brown and Walter (1990) take it a stage further. They claim that there is a direct link between problem posing and problem solving in that problem solving produces problem posing. English (1997) investigated the problem posing abilities of third grade children and found that they had a limited range of problems posed which did not change greatly with their participation in a problem posing program. The program did, however, improve their problem posing skills.

In a paper presented at a previous MERGA conference (1995, cited by Ellerton and Clarkson, 1996), Stoyanova suggested a three-way categorisation of problem posing under the headings *free*, *semi-structured* and *structured*. A *free* problem posing situation is one in which the learners are asked to pose a problem from a given situation or for a particular purpose. A *semi-structured* problem posing situation is one in which 'students are given an open situation and are invited to explore the structure and to complete it by applying knowledge, skills, concepts and relationships from their previous mathematical experiences' (Ellerton and Clarkson, 1996, p. 1011). A *structured* problem posing situation is one in which the learners are asked to pose problems based on a specific problem. It was this third category of problem solving that the researcher explored.

Observation has indicated to the researcher that many teacher education students are not enthusiastic about mathematics as a subject and problem solving in particular, and, in fact, experience a great deal of anxiety over the subject. This may be because of previous lack of success in mathematics and the tendency to rote learn, a method which is not as successful in problem solving as it might appear to be in other areas of mathematics. It may be that this lack of success in the past has conditioned students to react negatively to any aspect of mathematics and has lowered their self-esteem, thus making it difficult for them to take the kind of risks that may be necessary in problem solving, This is particularly true, it is felt, on the part of girls and women, and so relevant for teacher education in that the majority of primary teacher education students are female,

Problem posing is seen to be more akin to the way in which women operate and learn (Brown, 1984) so extending problems to be solved into the task of posing problems should suit female learning patterns. The researcher, accordingly, considered that an exploration of the effects this strategy, i.e. extending problem solving to problem posing, would have on beginning teacher education students, would add to the knowledge already growing concerning problem posing. The researcher explored the complementary procedure of problem posing with a view to ascertaining whether experience and hopefully success, in problem posing would assist students in problem solving. While, in general, problem posing can cover a broad range of possibilities, it was thought that, initially, this study should be confined to one of the simplest forms of problem posing, and that is, the posing of similar problems to given problems. This approach was taken because it was felt it would suit students with little or no experience in problem posing by not expecting them to stretch themselves too far. Because of the problem given, students would have a model and a structure which they could use in posing a new problem. It was on this basis that this study was structured, subjects were sought and the instrument devised and the research plan implemented in an attempt to explore the research question.

The Research Question

Will experience and skill in posing problems related to a given problem help pre-service teacher education students become better problem solvers?

The Subjects

The subjects were eight pre-service teacher education students in their first year of training. Volunteers were sought, as the group was large in number and it was considered that those who wanted to be involved would provide the most useful information. Twenty volunteers were sought but only fourteen students agreed to take part after they had been fully briefed on what was involved. Of these only eight continued through the whole process so the numbers fell far short of what was thought to be needed for a worthwhile study. This drop off possibly occurred for several reasons. One was pressure of work for students who were feeling insecure in some of their subjects. Another was that the study extended over two semesters and in the second semester, the researcher was not involved in the subjects they were taking. This meant that keeping contact became very difficult. Subsequently, however, the information gained from the participation of the students, even those who did not complete the sequence of problems, proved to be valuable and therefore analysed for comment. For this report, however, data concerning two subjects only will be presented, they being the ones on whom the most complete information is available.

Of the students who volunteered to participate, only two felt confident about their ability to do mathematics. Most were very anxious about the subject and several gave, as their reason for volunteering, their hope that the project might make them better at mathematics because they realised they would be teaching it. Their background differed, too. Only one had completed Level 3 in the Higher School Certificate and several had not done mathematics since they were in Year 10 or its equivalent in the case of mature age students.

All the volunteers were female. Only one male showed an interest in the project but did not eventually take part. The lack of male participation was not surprising though it was disappointing. The overwhelming majority of teacher education students in that year's intake are female.

The Instrument

The instrument consisted of 8 problems selected from *Teacher Tactics for Problem Solving* by Stacey and Southwell (1996). Three problems were of a general nature, three were numerical and then there were two that involved spatial concepts as well numerical concepts. Two additional questions were added at the request of the students who expressed a desire to have more practice in geometry. The problems were fairly elementary but lent themselves to good extensions and more than one strategy or answer, thus giving students the opportunity to demonstrate higher order thinking. Only one problem could be classed as a routine problem in the sense of requiring only the application of an algorithm while a part of one other problem was routine. The rest of the questions were non-routine. Also, all except parts of two questions required a number of steps for solution. The problems used are summarised in Table 1.

Table 1. Characteristics of Problems

Problem	Type	No. of Steps	Topic	No, of Possible Methods
1(a)	routine	2	money	1
1(b)	routine	3	money	1
1(c)	non-routine	-	money reverse	2
2	routine	3	money	2
3	non-routine	-	party tables	4
4	non-routine	-	tournament	3
5	non-routine	-	toaster	2
6	non-routine	-	chocolate bar	2
7	non-routine	-	cafeteria roster	2
8	non-routine	-	strip of paper	4
9	non-routine	-	diagonals of polygons	4
10	non-routine	-	exterior angles of polygons	4

Methodology

Each participant was issued with a small exercise book in which the first problem was printed. They also received a printed copy of the task to be carried out each week in relation to the given problem. They were asked to divide a double page in their exercise book as follows in Figure 1 to carry out their weekly task:

Week	Problem Solution	Problem Posed and Solved	Comment

Figure 1: Layout for Weekly Assignment

The task consisted of:

1. Solve the problem.
2. Pose a similar problem and solve it.
3. Write a comment on the problem given, the problem posed, or any other relevant matter.

This procedure was explained carefully to them in the first meeting between the participants and the researcher. Students did need, however, frequent reminders concerning this procedure and these were provided by the researcher in writing.

The participants were asked to hand in their book at the lecture each Monday at which time they would receive the next problem. This worked fairly well during the first semester of the year but when the researcher no longer took them for the lecture, it was more difficult to monitor progress. The books were marked each week and appropriate comments and suggestions were made. They were then photocopied and handed back to the student. This process continued for the duration of the study.

After participants had been working on the problems for four weeks, each one was invited to an individual interview with the researcher who had a prepared list of questions to ask. Some participants preferred to write their own answers so they would have time to think about the questions. Others were content to answer the researcher's questions so that she could record them. This process was repeated with the same set of questions at the conclusion of the study, i.e, when the participants had completed 8-10 problems. On one occasion, two participants asked could they be interviewed together and this was done. The questions asked were:

1. What is problem solving?
2. What did you find easy and what was hard about the problems you have been asked to solve?
3. Was having to pose a related problem a help in actually solving problems?
4. What use to you was the commentary you wrote on your problem solving and problem posing?

Analysis

The data collected consisted of the participants' solutions to the given problems, the problems posed and their solutions, the comments of the students each week and the responses to the two questionnaires. These were all analysed by the researcher and common elements extracted. Very different views were also noted and reasons for these sought. Particular attention was given to problem solving strategies and any changes in approach or skill that took place over the time of the study.

Results

The results are presented for two subjects in the study. Each section is described according to the data collected and then an attempt to provide a synthesis of them is made. As indicated above, the students were asked to solve a problem each week, pose a similar problem, solve that, and write a comment on the procedure.

Joyce: Joyce was able to get the right operation and answer for the routine question and for the routine sections of one other question but did not set out her work in a logical fashion. When she posed a similar problem, she changed the context and the numbers and, in one case, kept the numbers in the same ratio. This is an acceptable method of generating a further problem from a problem (Randall and Lester 1982). For the non-routine questions, Joyce used either a diagram, made a list or listed all possibilities as her main strategies and sometimes used two strategies together. On one of the problems she arrived at an answer which was obviously unrealistic but there was no attempt to check it. On another two problems, Joyce either misread the problem or did not examine the problem to see what it was really asking. She did admit that she was not always sure what was wanted. In regard to the first question, which had a non-routine third part, Joyce was able to pose a similar problem but admitted that after she had given a solution, she looked at it again a week later and changed her answer, at the same time saying in her commentary that she was desperate. Another indication of her lack of confidence was her several false starts and finishes. Work was crossed out fairly frequently and answers or methods questioned. She did, however, on the fifth question, begin to play around with the problem statement and explore some different interpretations and strategies. She only attempted to pose similar problems for two of the non-routine problems. This was unfortunate in light of the objective of this study.

Joyce's commentary on her work was reasonably full and apparently frank. On four of the problems, she indicated that she had either changed her thinking about the solution or would have liked more information. She also wrote some comments which were quite self-deprecating. In relation to the first question with the third non-routine part, she finished with "I can't believe I didn't see it earlier". This remark is even more poignant when one realises that she had not solved the problem after all. Another comment was "I don't like this much" then she questioned the use of the type of toaster in the question. Her solution (also expressed by other participants in the study) was to buy a popup toaster. On three of the problems she made reference to the usefulness of the problem. This seemed to make their solution more desirable and therefore she seemed more motivated to attempt the solution.

One other critical characteristic that emerged from the commentary was Joyce's preference for visualising through diagrams or writing lists or matching possibilities.

Answers to Question 1 in the questionnaire changed a little in the course of the study. From "finding an answer to something quite difficult" to "working out the answer to something that is not obvious at first" seems to imply that Joyce's view of problem solving changed from the main characteristic of difficulty to that of unfamiliarity or some sort of barrier. In the earlier interview, she pointed out that if there is an easy answer to a question, there is no problem.

Joyce stated in the first interview after attempting four questions that, with the exception of the third part of the first question, she found the questions relatively easy. She admitted, however, that she had trouble 'seeing' the problem. This was an interesting statement in that she found the correct answer almost immediately, then experimented with several other possibilities and finally opted for one which did not fit the specifications of the problem. In the later interview, she simply stated that any problem for which she could use a diagram, she found easier than the others.

Posing problems initially was not perceived as being helpful. Joyce did state, however, that if she was to do the task again, she "could see this might be of value" and she would use the tactic if she reached a standstill. This means she saw problem posing as a means of solving a problem but not necessarily as a strategy for her to develop her problem solving ability. In the later interview her comment was simply "helped to clarify my own thinking".

Joyce's comments in reply to the fourth question were as follows:

"I loved having a chat with you, B..... and it did help me clarify my thinking".

"It was good to be able to let you know how I felt about the question and explain how I went about it. I was more forthcoming in the comments because I knew and liked B..... Feedback is good and it is very important for me to read it".

Matilda: Matilda proved to be a much more confident and adventurous problem solver. Right from the first question she tried several different strategies for the non-routine questions. For the routine questions, she was able to find the correct answers without difficulty, set her work out logically, explained why she did each step and posed problems using a different context and different numbers. She attempted algebraic strategies even for routine problems and checked her work in several problems. She used several strategies in her solution of non-routine problems, including logical deduction, diagrams, listing all possibilities, making lists and looking for patterns. The problems posed were either using a different context or extensions of the original problem to develop generalisations. In one case, the same context was used but different numbers.

The commentary written by Matilda was more often than not interposed between steps in the problem solution. It contained frank admissions of frustration at least once and failure on two occasions. "There must be a way". Most of the other comments had to do with Matilda's thinking process and further clarified how she arrived at solutions, particularly where generalisations were concerned. The posed problems were all solved and in several cases led to further generalisation. She also suggested the solution to the toaster problem was to buy a popup toaster. It appears that the practicality of making toast got in the way of the mathematical problem this question posed.

Matilda's idea of problem solving was expressed in quite different terms from one interview to the next. In the first interview, she said "'Problem' has a negative connotation to me. Before I read it, I always think it will be hard just because it is a problem". In the second interview, her words were "Problem solving is finding an answer to something that is hidden at first - something you can't see as soon as you look at it. If it is a problem for one, it may not be for another". Like Joyce, Matilda recognised the relativity of problems.

In answering the second question on the questionnaire in the first interview, Matilda said she found the third part of the first question hard but the others were easier. She had spent a long time on it. She also said she used algebra "more than [I] probably should". In the second interview, her response was: "At the end they were easy. I got better. I found things hard at the beginning. I learnt how to think what it meant, how I thought about it. Once I knew what to do, doing it was easy."

Making up another problem helped Matilda mainly by confirming that she had done the first problem correctly. This was Matilda's response to the third question on the questionnaire in the second interview. This view seemed even stronger in the first interview in which Matilda responded by saying: "I find that posing another problem always helps me. It verifies that problem that I have already done and gives me more confidence for the next one." It may be that the boost in confidence was taken for granted by the time Matilda reached the eighth or ninth problem.

The commentary was also considered helpful. "It makes me think about what I am actually doing. How I am exactly going about solving the problem. Feedback is definitely necessary". This view was re-inforced in the second interview in which Matilda also commented on the value of talking to the researcher in the commentary or by phone and knowing she would always respond.

Discussion

The two teacher education students whose work and understandings have been reported are both mature aged women with different personalities and mathematical backgrounds. From their own reports through the interviews, both gained from the experience of posing problems similar to given problems, though one less so than the other. In this regard, it would appear that the process is better exploited by individuals with some degree of mathematical knowledge and confidence than by those with a less adequate mathematical preparation and less confidence.

Both subjects used a number of strategies, again with a differentiation between them according to their backgrounds and personality. More strategies were used by the students with the more adequate background. As neither had previously had any formal work on problem solving in their university course, it can be assumed that the strategies used were natural to them. Two further aspects of the strategies used relate to the content of the problems themselves. Problems that students perceive as having a practical use appear to be more worthy of solution in the eyes of the students and therefore they are more motivated to find a solution. In one case, however, the subjects' previous practice hindered their solution to the mathematical situation portrayed in the problem. The other relevant factor was whether a strategy that involved some form of visualisation such as a diagram could be used as part of the strategy. This preference shown by one subject in particular may be an aspect of learning style or it could be that the problems chosen all had easier solutions using this strategy.

The kind of relationship developed between the subjects and the researcher indicates that students respond well to the opportunity of being able to discuss their solution attempts and their frustrations with a person who can be relied on to listen and respond. This person could be a teacher but could just as well be a peer. The discussion can be spoken in person or on the phone, in writing or some other way. In this study, most of the conversation was through the interviews and through the commentary written by the subjects to which the researcher responded. The opportunity to record thought processes and feelings as a result of solving a problem proved helpful to both subjects. They gave two main reasons. These were that it clarified their own thinking and they were able to explain what they had done. It would be foolish to think that all students would find the process of recording their thoughts and feelings when they have solved a problem easy or helpful. Many would not and so it is necessary to develop some techniques which provide the incentive and the practice students needs to enable them to use this commentary effectively. Some possible techniques are:

- Ask orally a very straightforward question on the problem then get the students to record their answer. (one sentence at first, then two sentences, etc.)
- Provide the students with a problem solution. Ask them to write down whether they agree with the solution and why.
- Devise a game in which students draw a problem from a stack of problem cards. Each student presents their problem and ask the rest of the class to solve it and write an explanation of their solution strategy. Start with routine problems requiring one step only and increase complexity over time.

Feedback from the researcher also seemed to be important to the subjects. This finding re-inforces the view that the more feedback can be given and the faster it can be given, the more helpful it is to the learner.

Conclusion

On the evidence provided by the performance of the two subjects described, it is possible that posing problems based on given problems can be a useful strategy for developing the problem solving ability of teacher education students preparing to teach mathematics in the primary school, more so with the more mathematically able students than with those with a less adequate background in mathematics. Other factors of significance seem to be the practical nature of the problem and the ability to use strategies which involve some form of visualisation. The relationship with a reliable discussant and the frequency and speed of feedback are also important.

The findings of this study would be enhanced by confirmation or otherwise resulting from a repetition of this study with an increased number of subjects. As well, this study raises several issues worthy of further study. These include the role and forms of motivation in problem solving and problem posing and the relationship between problem solving, problem posing and personality. As problem solving is an important goal and process of mathematics education, these and related matters need urgent investigation.

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Teachers building number sense amid the challenges of change: Some case studies

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The paper describes some preliminary findings from a project in which teachers from two primary schools were observed over one school term as they implemented their own plans for encouraging number sense in their classrooms. Ownership of the entire project was invested in the teachers involved. Comment is made on some of the impediments to change and on one teacher as she implements mental computation activities.

This paper will describe a project undertaken in two schools in Perth during the time period July to December 1997. Analysis of data is still at a preliminary level and only some of the initial, tentative findings are described here. The paper will comprise three parts, namely an overview of the project, issues of professional development and finally outcomes of number sense development with teachers in the project schools.

The Project — Developing Number Sense in Education (DENSE)

The aims of the project were to:

- provide data from an in depth study of classrooms in which teachers have consciously introduced number sense approaches;
- evaluate a professional development model which empowered teachers to design and implement classroom change related to number sense.

Two Perth metropolitan primary schools were invited to join the project, one staff member of each school had had contact with the researchers on a previous project. Interested teachers within the schools attended the initial meetings. From these meetings three teachers and one coordinating teacher from School A and four teachers and one coordinating teacher from School B became involved in the project. The classes were almost parallel in age grouping from each school — two grade 3, one grade 4/5, one grade 5, a grade 7 and a grade 5/6/7 combined. The Developing Number Sense in Education (DENSE) project commenced with a meeting at the end of Term Two (June) and took place during Terms Three and Four (July to December 1997).

Data Gathering Methods

As the project was descriptive in nature qualitative data gathering techniques were employed, namely teacher journal writing, audio taping of group meetings, field notes and lesson observation notes by the researchers. In particular each teacher had a small audio-cassette in which she was encouraged to record her thoughts at any time. These tapes were quickly transcribed and sent to all teachers. Initial analysis is being undertaken by means of the NUD*IST computer program to identify emerging themes and issues.

Professional Development Model

From the literature on professional development one can loosely categorise models as, first, approaches which transmit new information to teachers via conferences, workshops and inservice work. Gordon and Tyson (1995) commented that this style of working has received much criticism in recent years because of its failure to foster long term change and to use teacher expertise. Secondly, there are approaches which involve the teacher and researcher in various forms of collaboration, usually a partnership and usually through an Action Research model. Reports of these projects suggest considerable learning by both partners and some more lengthy change taking place (Cobb, Wood & Yackel, 1990; Ellerton, Clements & Skehan, 1989).

Gordon and Tyson (1995) commented that the important thing which was missing from both of these models was an emphasis on the role of teachers as the decision makers in setting the agenda. In fact professional development often holds an assumption that the developing, reforming or empowering is done by someone else—not the teachers themselves.

Some initial research by Herrington, Sparrow and Swan (1995) attempted to use a model of professional development which considered an emphasis of teachers setting their own agenda. The mathematics case methods model (Barnett, 1992; Barnett & Tyson, 1993(a); Barnett & Friedman, 1995(a); Gordon & Tyson, 1995) has an expectation of teachers recognizing themselves as their own change agents. These projects provide evidence of a third category of professional development model. Essentially, in this model, the program originates in, and is driven by teachers' concerns, interests and the realities of their classroom. In that sense it is a model of teacher empowerment in which teachers identify and meet their own needs for professional development. Robinson (1989) discussed the empowerment paradigm and noted the shift from teachers being manipulated by others into changing to one of personal control of professional development. This represents a move from an instrumental view of teachers to one of teachers as responsible professionals.

The Process of Change

Clarke and Peter (1993) have noted the processes of change in teachers' practice, knowledge and beliefs are intertwined and form part of a learning process. They suggested two central components of professional growth:

- teacher experimentation, and
- teacher reflection.

Bearing this in mind and the earlier comments on the inadequacy of the transmission models of professional development in establishing long term change, it would seem more appropriate to consider models that involved the teacher directly and overtly in the process. One will need to create situations where beliefs are faced and considered. This demands powerful interventions which challenge, and yet are safe situations in which teachers can take mathematical, emotional and intellectual risks (Wilcox, Schram, Lappan & Lanier, 1991). The empowerment paradigm sees the change process as situation specific and cannot be seen as an attempt to implement a specific reform or apply a specific change model to a school generally. Change and professional development, Robinson argued, should be seen as an educative process with no universally correct and acceptable end point to be achieved. Teachers are given information about the range of choices available so that they can make meaningful choices for their classroom.

Work by Ellerton, Clements and Skehan (1989) and Herrington, Sparrow and Swan (1995) used the Action Research Cycle with an emphasis on the *insider* (teacher) rather than the *outsider* (researcher, curriculum leader) setting the agenda for discussion and development. The Action Research Spiral of plan, act, observe and reflect was central to the work and was designed to establish the need for reflection and experimentation by the teacher and to set the problem in the specific and real classroom.

From the models of working with teachers in professional development the Action Research Cycle embodies the reflection and experimentation aspects necessary for change (Clarke & Peter,1993) and where this can be coupled with more natural teacher methods of narrative (Johnston, 1994) it should support the reflection-on-action aspects of teaching. Schon (1983) has used the term *reflection-in-action* to describe this continual process in which teachers are engaged. He also noted the idea of *reflection-on-action* which is carried out after the event and is more able to be assisted by a collaborative process. The DENSE project used both models of reflection with teachers so that they could broaden their own content and pedagogical knowledge base related to number sense and its development in the classroom.

Emerging Issues from Professional Development

Impediments to Change

The major issue emerging is concerned with change and the impediments and challenges felt by the teachers trying to implement change in their classrooms. The impediments impacting on the teacher paint quite a complex picture. Even where there is a willing teacher, the effort to change and develop classroom practice beyond a traditional form is countered at almost every stage.

Expectations

In itself, this is quite a complex constraint being constructed of a range of sub issues. One might argue that some of these are 'felt issues' rather than 'actual issues'. The 'felt issues' being derived possibly from the social heritage of the staffroom (Mousley & Clements, 1990). Generally they are not tested in reality i.e. actually asking the parent body if this is their expectation of the teacher and the school. Expectations emerging were attributed to parents, other teachers in the school and teachers from the high school which the children would attend. The expectations issue was especially strong in the comments of the Year 7 teacher.

> *I'm in a difficult position really, having Year Sevens, it's the year before they go to High School, and unfortunately there is an expectation of what a Year 7 student should know.*
>
> *I'm also very aware of the expectations of parents, other Year Seven teachers and the High School next year,... but I have to tread very carefully, or a lot of parents will chop my head off.*

Concerns were evident related to the expectations of other teachers in the school and to comparisons which might be made between classes.

> *... but I'm doing this bit of maths this way, but the other Year Fives are not doing it this way, and if you compared all the Year Fives, how are you going to compare them, are you going to compare them by saying, here's a sheet of algorithms go for it, and the other lot of Year Fives will get them all right and mine will get them all wrong, but they'll know why they got them wrong.*
>
> *I think where P. feels a little bit threatened is that she's come in new to the school, and these teachers have been teaching here for a long, long time, and they feel that they know how to do it.*

The expectation of a traditional way of doing mathematics is strong and exerts a pressure on teachers to conform. One needs to be quite confident in oneself and in mathematics to be able to move away from the normal because as soon as one does change one is open to criticism from a number of sources.

Time Factors and the Syllabus

A feeling emerging from the data was one of the teacher being presented with yet another thing to fit into the finite school day. There did not seem to be an alternative argument, for example, that if something was good and had to be put into the day then something judged less important would have to go out of the day to make room. An image of the teacher trying to do all things but not giving any sufficient coverage to be effective began to emerge.

> *Yes it's true, you've got to get these kids through the syllabus.*
>
> *I think at my last school, it was very much sort of, this is what's in the syllabus, this is what the Headmaster comes in and wants to talk to them, they need to know the things they are supposed to know according to the syllabus.*
>
> *You know it's really good doing all this number sense but I still have to do long multiplication*

It appears for many teachers that syllabus coverage is paramount. The quality of the outcomes, in terms of the children's learning, was only a minor concern. It is interesting to note that number sense was seen as extra to the syllabus, the latter emphasis being, one might surmise, in the eyes of many teachers, to be concerned with mastery of the standard methods of computation.

The Children in the Classrooms

Comments from the teachers related to the children often seemed to have a negative spin to them. Typical would be a comment that pointed out what the children could not do and this was offered as a constraint to doing whatever was intended.

> *Especially when it's Year Seven when you want them to give you how they've come to their answer, if they've had that conditioning [years of formal algorithms], and one of the things I found teaching our Year Sevens was like, there's your answer, why do you want me to give you any more information?*
>
> *... and because they have not had all this background in Year One and Two, what we've done in Year Three has virtually been back down in Year One and Two.*

For many teachers the combination of low self confidence, unresponsive children and loneliness, if they are undertaking a new way of working without the support of the project, makes for a non-starter and the ideas are not even tried. One part of Clarke and Peter's (1993) model for change does not take place.

Available Resources

Discussion often noted the constraining effect of resources that were available, notably the set textbook for the class.

> *... and the books that they've got are Boomerang Maths books, which seem quite OK. But I mean, it's not what I would have chosen necessarily, but what I've got to try and do is to sort of justify the people buying this book, and try to fit it in with the way I'm doing maths and the content that we, that we're covering.*

The text book, like the earlier factor of expectations, can impede change. Most textbooks promote a model of computation which emphasises the standard methods of computation and instant recall of basic facts. In many schools there is also an expectation that teachers will teach to the book and that each year will complete their designated book. Again this does not give much time to do the extras such as number sense.

The Teachers' Background Knowledge and Beliefs

Throughout the data there is comment related to a new awareness in mathematics teaching and in particular to the idea of number sense. There is, however, still a recurring tension between this and the almost ingrained beliefs and ways of doing things. Tradition and the fact that "everyone knows how a teacher operates" from their apprenticeship of observation as a child seem to be major factors in this dilemma.

> *And it's confidence, you know, we are such a structured profession,... and it's very hard to break away from that, and even though I've been teaching, you know, for less than ten years, I came into it knowing that I had to do it [teach in a structured way]*
>
> *I have to staple my tongue to the roof of my mouth sometimes to stop myself from telling the kids...*

For some of the teachers there is a reluctance to embrace the new idea of number sense due to their own felt inadequacy with mathematics content and understanding.

> *I think I get muddled up because I sort of anticipate, I sort of have in my head what the kids are going to say and how they are going to respond, but if they come up with something different, it sort of throws me a bit, and I'm not clever enough sometimes to sort of use what they say effectively.*

Without the support of the project this would be the main challenge to any adoption or even experimentation with number sense. Everything about mathematics before had been clear-cut and predictable. It is a major shift in teaching style for many teachers to leave a question open and invite a range of often unpredictable responses with the distinct possibility that at least one will be different from the predicted route.

Emerging Issues Related to Mental Computation

Description of a Mental Computation Lesson, Kaye, Year 5 Class

During the project one of the investigators sat in on a number of number sense lessons at his request. His intentions were twofold: first, this would provide the only direct evidence of what was actually happening in the project classrooms; second, it was hoped that presenting teachers with a detailed account of what the investigator saw and heard, together with some comments, might form a useful form of professional development. Here is a description of one lesson, in the form it was later given to the teacher.

11.15am. Class on mat at front. Explanation of strategies for 89+ 26: Kaye describes pictures of three children [shown calculating 89 + 26] on the overhead. 'Imagine you are the girl, think about the answer to 89 + 26 and how you could describe what you did to somebody else.' Kaye has a large sheet of brown paper attached to the blackboard by the OHP picture. It is marked 'How can we work it out?'

After a pause, Kaye asks children to explain what they did. She writes each explanation in words and symbols on the brown paper.

- 8 + 2 = 100, 9 + 6 = 15. 115. (When asked why 8 + 2 = 100 the girl retreated into silence.)
- With fingers. Did 89 add two tens and 6.
- Did it by fives. Added 8 + 2 = 100, then 9 + 6 = 15, added 100 + 15 by 5, 10, 15. 80 + 20, 9 + 6.
- Imagined the numbers written down vertically, did as a written sum.
- Added 8 + 2 then added a zero, then 9 + 6. 'How did you know to add a zero?' Silence from boy.
- Changed it to 86 + 92 and got 115. [Did he mean 86 + 29? This was not resolved]

11.35am. Discussion of Strategies: 'Let's have a look at some of these strategies.'

- Discussed why 8 + 2 = 100, in terms of 8 tens and 2 tens. The child's method was validated.

- Discussed a quicker way of adding 20 to 89 than counting in ones using fingers. 'Count in tens.' 10, 20, 30...' 'Count in tens from 3.' '3, 13, 23...' Count in tens from 9.' 9, 19, 29...59.' Count in 9s from 89.' [More hesitantly] '89, 99, 109...' This felt like a breakthrough for some children.

11.43am. Extension of Activity: 'Before we do some more, would you rather do that calculation [89 + 26] with a calculator, in your head, or on paper?' All three methods were advocated by different children. 'Can someone give me a calculation they would need a calculator for?' '4 billion + 561', 5230 x 1000' were suggested and queried.

11.48am. Activity 2: Ten calculations were revealed on the OHP. 'Which of these could you do in your head?' Each was considered with children being asked to justify their choices.

11.55: Activity 3: 'This example 88 + 88 + 88. Work it out mentally and then write down the answer and explain how you did it.'

12.06 - 12.15pm. Discussion of Activity 3: Several children were asked to share what they had written.

Issues Arising from the Lessons

A mental computation 'lesson' has traditionally been envisaged as lasting about 10 to 15 minutes. In contrast Kaye's mental computation session lasted the entire 60 minutes of the mathematics lesson, although Kaye admitted she had some reservations about this.

> *I would never do this for so long usually and I would prefer to do it with a small group while others were working*

Again, a traditional mental computation lesson would consist of a series of short computations to which children write answers in silence, emphasis being placed on speed and accuracy. In contrast, Kaye's lesson involved a great deal of class discussion with an emphasis on encouraging children to explain their strategies and justify their answers orally.

> *I emphasise children's explanations, children listening to the explanations of others, the fact that there are different ways of mentally doing the same calculation, the value of giving your own idea 'even if you think someone else might think it silly', giving children time to think when I ask them a question.*

The purpose of a traditional mental computation lesson, if a purpose was considered by the teacher, might be to increase knowledge of basic facts or to sharpen up the children before the 'main' part of the lesson. Kaye's reason for emphasising mental computation is clearly different.

> *The area of number sense that I'm dealing with, with my Year 5 class is Developing Mental Strategies. And the reason I've chosen this is that it seems that children generally have a poor understanding of basic facts and place value. And when they're confronted with a problem they're easily stumped. They don't appear to have a variety of strategies in place to work it through, and even those who appear to be fairly well skilled at pencil and paper computations have trouble when they're confronted with a problem that's posed in different ways.*

When reflecting on the value of her mental computation lessons later, Kaye felt that these aims had been realised:

> *Having a concentration on that and just forgetting about the other parts of number maths for the time being, I'm really amazed how very quickly those children who just had absolutely no idea, I think the first session that we had Alistair was here and there were kids who were just gazing into space, absolutely no idea about how to go about doing that, and other kids who were more able were able to*

> *share their strategies, and those children who had no idea have now got strategies to begin to use, and they are choosing other people's strategies, and with some success, which is really terrific.*

However Kaye also saw them as having much broader benefits still.

> *... but the really terrific spin off is that [their communication skills] have just improved out of sight, as they've realised how precise they have to be in their oral sharing for someone else to understand, because I actually get the person who's listened to someone sharing, they then have to tell that strategy to somebody else and they have to listen very carefully to someone else's strategy and then explain that to another person, so their listening skills have improved, their oral sharing skills and their written skills because they know how precise that has to be for someone else to read it.*

Finally the written record of the lesson, while being very informative for the investigators, proved also both valuable and reassuring for the teachers.

> *I guess copies of the lesson observations are the most useful in terms of perhaps learning teaching strategies used by others. I'm particularly interested in the other Year 5 class at Ashburton, especially as we have a common area of focus.*

Interim Conclusions

Implementing a research and curriculum development project in this way has a number of difficulties. From the curriculum development point of view, one has sometimes to follow the teacher's advice and 'staple one's tongue to the roof of one's mouth'. Things can appear to move slowly, erratically, and in unexpected directions. If teachers follow their own intentions, they will not accord in every detail with that of the project leader and director. On the other hand the process generates a much greater head of steam because the teachers, having chosen to take part, stop looking for directions and use their own judgment. And the exciting thing is that they constantly reveal what a wealth of common sense, sound judgment, and educational creativity they possess.

From the research point of view, the reliance on audio-cassettes for most of the data collection proved to have advantages and disadvantages. The main disadvantage is that teachers are, on the whole, much better doers than talkers. They will spend hours planning and preparing lessons, but proved somewhat loathe to spend five minutes on reflections and afterthoughts (though almost all said that they spoke into the cassettes much more than they would have communicated on paper). The great advantage was that in several cases teachers spoke into the cassettes (often late at night) with immense candour with a flavour and intimacy which would not have come through on paper. We conclude with one such glimpse.

> *I think this whole project has led me to rethink a whole pile of things to do with my own teaching and I think that's a good thing. I don't think I was doing a bad job before, but I think it's made me [give] children back the ownership for their learning, instead of worrying all the time that I haven't taught this and I haven't taught that, now I can give them the opportunities to learn things that they need to learn, and that seems to be spreading right across the things that I do. You know, I've been teaching for 23 years, you get a bit stale, so sort of a new approach is very elevating, stimulating for the children. ... I'm sort of almost sad that our part in this has come to an end, but I won't end what I'm doing, I shall keep on going and I shall make sure that I continue to find new things to do, maybe I'll move on to something other than mental computation, I'd like to develop*

> *[fractions] a bit more, I don't feel so worried about that now, I used to think God, fractions, what am I going to do with this! But the kids are sort of feeling quite chirpy about what we've done, so I think maybe over the holidays I'll think about where I can go with that.*

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The *invisible wall* project on problem solving processes: Concepts and methods of interpretive work with high-resolution data

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The project *pupils work on problems with a goal which cannot be reached* ("*invisible wall project*") analyses problem solving processes of children in grades 3-4 and 7-8. So far, we have described basic components of problem solving abilites by using interpretive methods. We now want to apply statistical methods, e.g., for comparing abilities of younger/older children. Consequently, we have complemented the qualitative interpretive methods by quantitative methods. The paper describes the methodological background of both sides of the project.

Introduction: The *invisible wall* project and it's methodological problems

Since 1992 I have been working - together with several groups of student teachers - on the analysis of problem solving processes. We focus on components of problem solving ability which are *not* subject of mathematics lessons. As a consequence, we did not deal with word problems and other Aclassical problems@. The central idea of our research was to use sets of tasks which are *unsolvable* which means they have a goal which can not be reached. The unsolvability, however, is of a kind which can be understood even by younger children (e.g.: try to find exactly 4 different numbers out of the set {1, 2, 3, 4, 5} which give the sum 9; see another example on the next page). Subsequently, we shall use the term *impossible task* as well. Impossible tasks are used for several reasons:

- The impossibility of the task acts as an A**invisible wall**@ for the pupils' actions: the pupil tries to solve the task, gets stuck ("bounces against the invisible wall"), and tries a different way to do it. As a consequence, we get information out of the pupils' attempts to solve the task.
- *The tasks stimulate natural reasoning behavior*: the pupil looks for reasons "why he/she cannot do it ..."!
- Because the goal of the tasks cannot be reached, they cannot be solved by chance. They are not too simple even for pupils who are good in mathematics. Consequently, a puzzle which is not solvable can have a far simpler structure than one which can be solved. This is *good for the transcription of problem solving processes*.

In the first part of the project we have identified elementary components of problem solving behaviour which are used by younger children (the methods and some of the results are described in STEIN 1997). The search for those components was organized as search for noticeable patterns in the subject=s behaviour.

In the present stage of the project we try to describe profiles of problem solving processes. As will be discussed later, our approach is *qualitative-interpretative*.

With the help of the above mentioned grant, we have protocolled and fully transcribed approximately 160 interviews with pupils of grades 3-4 and grades 7-8. This material should be open to analysis by *quantitative* methods.

Qualitative methods have a high demand of manpower (for transcription and interpretation). So, collecting large numbers of quantitative material for making it accessible to statistical analysis demands careful considerations:

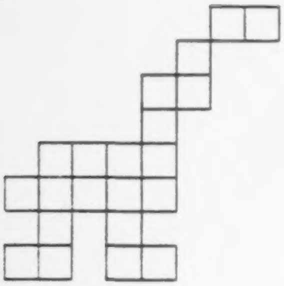
- Since transcription is time consuming and causes high expenses, transcription techniques should be not more complex than necessary. On the other hand, transcripts should comprise enough information for substantial interpretation.
- Editing the collected material for statistical analysis should not wipe off the interpretive features of the research.

This paper focuses on the solution of the *methodological problems* described above.

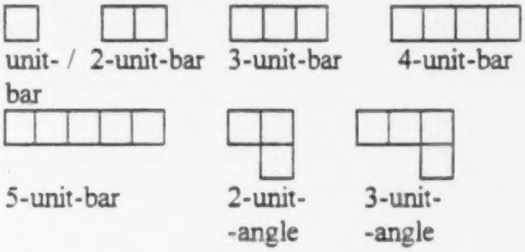
The tasks

We work with a range of different puzzles. In this paper we concentrate on the following puzzle:

"Giraffe"



The puzzle shall be filled exactly with the following parts (each is given once).



Control of Complexity as criterion for the choice of document type

One main *technical* problem in designing an interpretative study lies in *controlling the complexity* of the documents which are to be analysed.

Videotapes are of very high complexity. The stored information bears a manifold of different aspects. If a researcher gives a videotape to different groups of interpreters he/she will find that they focus on different aspects. Even if he/she defines strictly the aspects under consideration he/she will still find that different groups will use different bits of information to back their interpretation: they may focus on body language, on intonation, on mimics, and so on. Even worse: the researcher cannot control subsconscious effects of body language etc. on the interpreters.

As a consequence, videotapes have no possibility of controlling their complexity. Audiotapes have the same disadvantages as videotapes.

Transcripts show a well defined part of the original video- or audiotape in written form. Writing a transcript, the researcher has full control over the complexity of his material. As *methodological consequence*, our analyses are based on transcripts.

Transcription

Methods of transcription

Within any project of interpretive research, transcription has to follow well defined standards. Any report on interpretive research has to give full information of the transcription rules.

Under the aspect of *control of complexity* the researcher has to decide carefully what aspects of the tape shall be written down. We find different stages in complexity:

- *Speech transcripts* contain at least an exact record of the words spoken.
- *Action transcripts* give additional information about actions performed by the pupils and normally contain information about speech as well. Special care has to be taken to define precisely what actions are transcribed, what actions will not be transcribed. The transcribers should stick exactly to those rules.

Methodological decisions of the project

Since we work with younger pupils, all our tasks use material which can be manipulated by the children. In our research we see pupils act (and react), and try to describe and classify the observed patterns in the problem solving processes.

The "spoken word" does *not* play a big role in our research, since pupils do not speak very much during their work (we do not encourage pupils to "think loud" since this would interfere the process of solving the task). For transcription of the spoken word we chose not to simplify or correct sentences. We measured *breaks* in seconds and included this in the transcript, for instance by [...] for a three second break, [10 sec] for a ten second break. Passages which could not be understood were measured and included in the transcript, for instance by (....) for 4 seconds of murmur which could not be understood.

Since the processes observed consist of actions, we use *action transcipts*.

It is useful to distinguish between *hard rules* and *soft rules*. We transcribe *every single action which ends with a puzzle piece laid down* and *everything spoken out loudly* (hard rule). Transcribers included information about body language or shown emotions if they considered them important (soft rule).

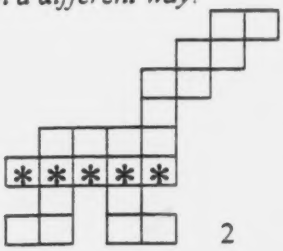
The inclusion of time-information is essential for action transcripts. We note - for each action transcribed - the exact time of its beginning and its end. Having exact information about time, the interpreters can consider the speed of action.

The following example (*2.05* means: we start after 2 minutes and 5 seconds of work) shows two pupils working on our puzzle. Transcription is as viewed "from above". That means, the pupils are sitting at a table with the puzzle between them. Text on the left (right) side refers to the pupil sitting left (right). Actions which are written below other actions happened later.

Interviewer: *And if you start in a different way?*

No, this has to be put there.
points to the 5-unit-bar

points to✱ *No, the 5-unit-bar*
must be placed there.
It has to be placed here!
points to✱



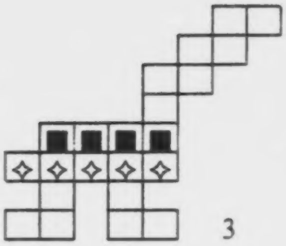
No, let us look everywhere 2.05
takes the 5-unit-bar into the hand

moves the 5-unit-bar over the
puzzle

2.17

puts down the 4-unit-bar
The 4-unit-bar has to be placed here

But there are no other possibilities for it.



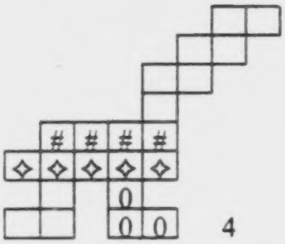
3

puts down the 5-unit-bar

No, thats not the only place.
removes the 4-unit-bar

2.23

points to the 4-unit-bar
This has to be placed there.
Points to #. We don't have possibility for this piece.



4

No, wait, lets first fill in this place.
puts down the 2-unit-angle
For instance.

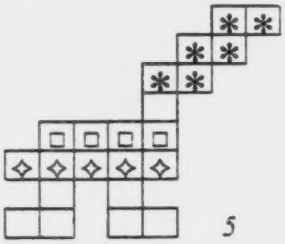
puts the 4-unit-bar to #

2.40

removes the 2-unit-angle

takes 3-unit-bar from his partner /holds it over*
this is not possible.

puts the 3-unit-bar away*No, this can not be done.*



5

holds 3-unit-bar in his hand

takes 4-unit-bar
and 5-unit-bar away.
It must be possible to solve it

Communicability of documents

Whatever the chosen type of transcript is, the researcher has to decide about the *resolution* of transcription. High-resolution-transcripts may contain valuable additional information about body language, intonation and so on. Especially in the case of high resolution action transcripts the researcher will find that they may not be communicable to anyone outside his/her research group. Such transcripts may be useful - but for *internal use only*. If high resolution is necessary for the aims of the study, the researcher should use low resolution versions of his transcripts for publication of results. Under the aspect of communicability, video- or audiotapes are not suitable documents to work with.

As the example above shows, we take care to keep our transcripts communicable to others. (Though written with a different focus, L. Richardson's paper on *writing* (Richardson 1994) shows the same interest in communicability.)

Interpretation of documents

Problems of interpretive work

Any bit of the analysed document can be seen under different aspects. Any remark or action can have a large *variety of meanings*. This is the reason why we speak of *interpreting* documents, and do not use terms like *classifying* or *grading*.

The applied technique of interpretation depends on research interest and theoretical background and should be described carefully. Though there are nearly as many interpretation procedures as there are research projects, some standards are common (e.g.: interpretations are done by *groups of interpreters; indicators* should be used to support interpretations, esp. in the case of conflicting opinions about).

Methodological decisions of the project

In our project, the behavior of the children is analysed under a broad range of aspects. In many cases there will be more than one interpretation of the same behaviour. The same action may be interpreted, for instance, as a consequence of a social conflict between the two children, or may be seen as influenced by gestalt operations or be understood as guided by explicit use of heuristic strategies. The variety of meanings is a very important reason to follow the standard of *working in groups* when interpreting a given document, since only a group of interpreters has the chance of avoiding one-dimensional interpretation.

If a group of interpreters intends to interpret a part of an episode unambiguously, they need *indicators*. Indicators may be *weak* or *strong* with respect to the amount of ambiguity they dissolve.

In our project, we want to see whether a pupil has "understood" the unsolvability of the task. A week indicator for this is the plain remark "*This puzzle is not solvable*". This remark *may* be a sign of a deeper insight, but as well it may just mean "*The puzzle is not solvable in the way we started to do it*".

Strong indicators for insight in unsolvability are (among others)

- refusal to continue work when asked to start a new attempt in solving the puzzle; o
- explicitly contradicting the other pupils' opinion that the puzzle is solvable.

Structuring documents

The problem of finding a structure

Very often we shall face the problem that the source document is too long for getting a full overview of its content. It will be helpful then to structure it by dividing it into smaller passages which are called *episodes* (Krummheuer 1992). Episodes may be named. They may be split into *phases* (Voigt 1984, p. 128).

When working with a full transcription of the source document, episodes may be used to give a short summary of the transcript.

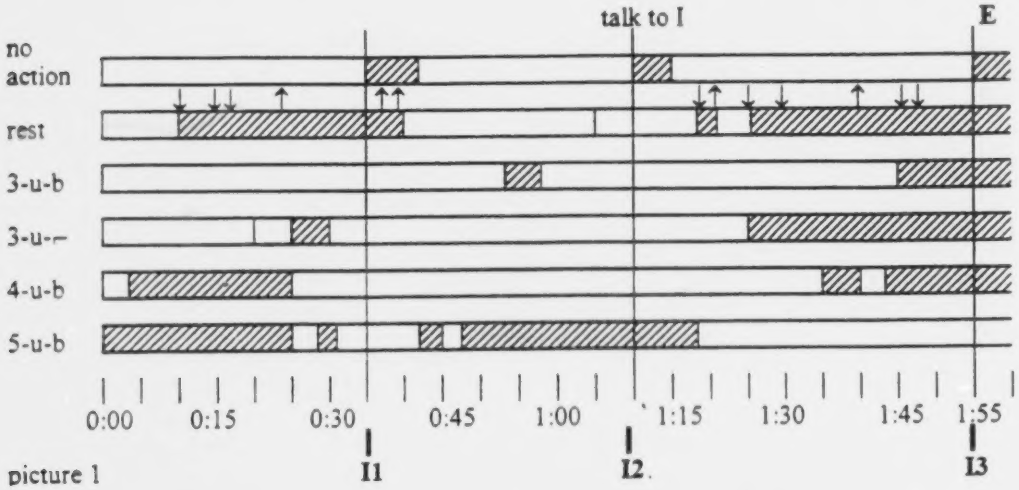
A so called *episode plan* (Wollring 1994) may as well be written *before* doing anything else. This episode plan can be used for describing the content of the whole document. Under the aspect of communicability, it is a valuable information for the reader of a report who may use it to locate the position of the transcribed episodes within the basis document.

Methodological decisions of the project

Action profiles: The example above shows that action transcripts consume lots of space. A 5 minute interview easily needs 20 pages of transcription. In our project, neither of these "episode approaches" for structuring the transcripts is useful: having a high number of transcribed interviews, we have to use coders for structuring the documents. Since this coding has to be reliable for later use in statistical analyses, the built-in ambiguity of the interpretive approach of phases and episodes is not helpful in our case.

The first answer to this problem are *action profiles*. In the case of our puzzle, we have "time bars" to code when the 5-unit-bar resp. the 4-unit-bar resp. the 3-unit-angle resp. the 3-unit-bar is laid down, and how long it stays on the puzzle. Another bar codes actions with the rest of the puzzle pieces. In the last case, arrows ↓ ↑ denote the moment when one of those pieces was laid down resp. removed from the puzzle.

I1, I2, I3 means questions of the interviewer, following the standardised design. The following picture1 shows the (for reasons of communicabiliy simplified) action profile of two pupils in grade 4. The explanation phase (marked **E**) is shown abridged. It begins at 1:55 and lasts until 2:10 with the pieces lying on the puzzle as shown.



picture 1

To help the reader understanding the profile, we comment the first 35 seconds of the profile:

The 5-unit-bar is laid down as soon as the work starts. It remains on the puzzle until 0:25. App. at 0:02 the 4-unit-bar is laid down. This piece remains on the puzzle until 0:25. After some hesitation, from 0:10 on the pupils start working with small pieces. (Normally this will be "filling up the head or the feet"). At 0:25, the 4-unit-bar and the 5-unit-bar are removed, the 3-unit-angle is laid down and remains on the puzzle for 5 seconds. Shortly before 0:30, the 5-unit-bar is laid down again. Then, the 3-unit-angle and the 5-unit-bar are both removed. There is no action from 0:30 to 0:35. Some small pieces are still lying on the puzzle. At 0:35 we find the first intervention of the interviewer.

We see that the combination 5-unit-bar/4-unit-bar - which is essential for understanding why the task is impossible - was only found at the beginning of the interview. Such problem solving processes are called *decreasing*. Normally, in such interviews children do not find good explanations for the impossibility of the task.

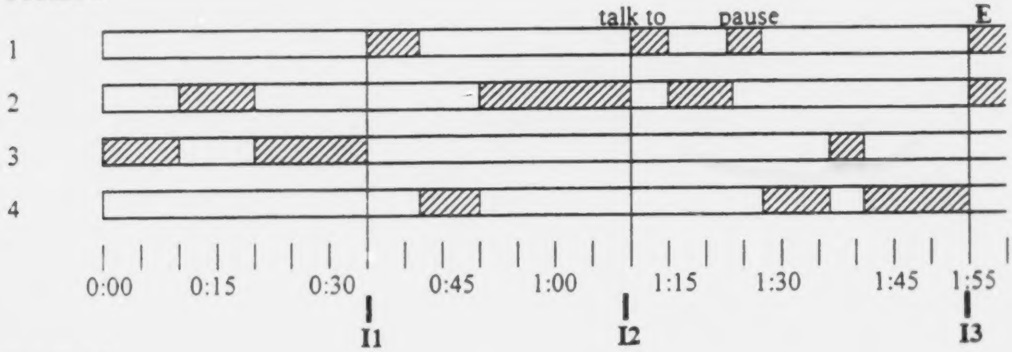
Structures of action profiles: It is quite obvious that it is not difficult to get highly reliable codings of action profiles. We are now able (though we can not do it in this paper) to describe different types of such profiles. For instance, the example above shows a decreasing profile. If the 5-unit-bar is lying on the puzzle during the last phase of the solving process only, we will call the profile increasing.

Using this classification, we will be able to use statistical methods for describing differences between problem solving abilities of younger pupils compared with pupils of grades 7 to 9. We might ask, for instance: Do older pupils produce significantly more increasing profiles than younger pupils?

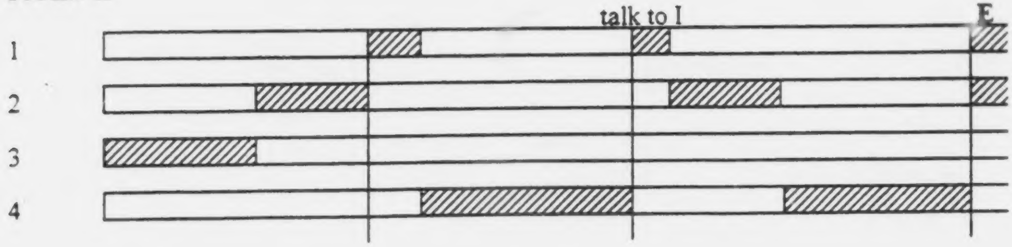
On the other hand, our action profiles will have to be "read" and "understood". We may ask, for instance: *increasing profiles end with the 5-unit-bar "in the right place". Is this in all cases an indicator for a "good" problem solving process?* So, in the long end, we will have to return to interpretive work. How this can be done, will be shown in the next sections.

Interpretation profiles: As in the case of action profiles, we work with time bars. Now, however, the bars stand for a qualitative interpretive analysis of the problem solving process. With our puzzles, we use the following categories:
(1) No action; (2) Sequences of *gestalt* reactions (there is some similarity to Davis/Mc Knight's notion of *visually moderated sequences*); (3) Sequences guided by logical or analytical considerations; (4) Other systematic behavior (trying to fill the puzzle, beginning at one end of it without leaving gaps, for instance).
The result is called *interpretation profile*. In the following, we see two different interpretation profiles for the action profile of the preceding page:

Profile I



Profile II



Under the aspect of ambiguity of interpretation, the interpreters tried to develop for the same interview *two* interpretation profiles which should differ as much as possible. We call this method *divergent coding*.

Though rather different, the interpretation profiles of the above process have some features in common:

- Both interpreters accept a procedure with a logical structure only at the beginning of the interview. But disagree about the length of it (ProfileI: 10";

Profile II: 23"). Anyway, we can conclude, that the "logical action" at the beginning of the work will not have been caused by some careful consideration.

- Both interpreters see some "other systematic" behavior at the end of the interview.

Altogether, the interpretation profiles give a strong feeling that the pupils may have started with "a good idea", but if it was so, this "got lost" afterwards. They did some trial end error in the middle phase - about the quality of the actions the coders have different opinions - and ended their work with no real insight in the impossibility, but were unsuccessfully trying to fill the puzzle up (not surprisingly, the action profile shows many actions with the small pieces). The explanation is characteristic: "It can't be done because there are some pieces wrong".

Conclusion

Our research interest led us to develop a high resolution transcription technique: *action transcripts*. Understanding the problem solving processes and answering the question whether the pupil "understood" the impossibility of reaching the goal, needs *interpretive methods* which bear in mind that nearly every action of the pupils can have more than one "meaning".

The basic methodological problem of our research is caused by the high number of interviews. High resolution transcripts are very long. We have to compress the information contained in them to make it accessible to statistical methods. On the other hand, we do not want to loose essential aspects of interpretive analysis of the interviews.

Our solution to the problem described follows for some distance a design which seems to be characteristic for interpretive research with large numbers of data: parallel or additional to interpretation processes there are methods applied for coding the data in a more efficient way. (For instance, the interpretive analysis of school lessons may be complemented by *Bellack coding*). Having standardised and analysed our data by using and categorizing our *action profiles*, we leave the standard pathway in the long end by adding a new element: *divergent coding* brings back the ambiguity of interpretive work and can add useful information to otherwise rather dry statistical data.

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The incidence of misconceptions of decimal notation amongst students in Grades 5 to 10

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This paper reports the results of a test of decimal understanding based on choosing the larger number from pairs of decimals. Ten incorrect ways of thinking about decimal notation are described. The testing of 2517 students from Grade 5 to Year 10 enables reporting of the incidence of misconceptions according to both a primary and a refined classification. Some variations from previously reported results are noted.

It is now well documented that many students throughout schooling and indeed many adults have difficulty understanding the notation used for decimal fractions. The recent Third International Mathematics and Science Study showed that internationally about a half of 13 year old students could select the smallest decimal number from a multiple choice list of five decimals (data held at the Australian Council for Educational Research). Similar results have been known for many years in several countries (Bell, Costello and Kuchemann, 1983; Carpenter, Corbitt, Kepner, Lindquist and Reys, 1981).

There are various ways of classifying the erroneous rules which students may apply when deciding which of two decimals is the larger. The coarsest classification is that some students select "longer is larger" (e.g., deciding 0.125 is larger than 0.3) whilst others select "shorter is larger" (e.g., deciding 0.3 is larger than 0.496). Sackur-Grisvard & Leonard (1985), Nesher and Peled (1986) and Resnick, Nesher, Leonard, Magone, Omanson and Peled (1989) refined this classification to three erroneous rules. Sackur-Grisvard and Leonard (1985) had identified erroneous rule use principally from a test where students had to place three decimals in order of size. Resnick et al (1989) modified this to comparison of size of pairs of decimals, and in some cases of a vulgar fraction and a decimal. Moloney and Stacey (1996, 1997) modified the Resnick et al test to exclude fractions as these seemed to cause extraneous problems.

Stacey and Steinle (1998) reported on interview data and an item by item analysis of an extended test of 25 decimal pair comparisons which lead to identifying further misconceptions about decimal notation. In total, ten patterns of thinking were identified. In the first section of this paper, we outline these patterns of thinking. Then we describe a new decimal comparison test which will classify students according to this new scheme. The distribution of a sample of students from Grade 5 to 10 according to this classification is presented. An indication of the variability of students' thinking patterns by school is presented which provides preliminary evidence that sometimes students misconceptions about decimals are a direct result of instruction. In other cases, misconceptions arise when ideas interfere with each other.

Thinking About The Size Of Decimal Numbers.

Stacey and Steinle (1998) used interview and test data to identify the following possible ways of thinking about decimal notation. The numbers of students found to be using each of these are reported in a later section. The detailed types of thinking have been grouped according to a coarse primary classification of students who make choices based on length (longer-is-larger and shorter-is-larger) compared with those who make more sensible choices.

Longer-is-larger Misconceptions

Students in this primary misconception category generally believe that a longer decimal is larger than a shorter decimal. It includes both "whole number rule" and "zero rule" as described by Resnick et al (1989).

String length thinking: These students judge only on length, effectively treating the decimal part as a whole number. They interpret the decimal as two whole numbers separated by a marker, with longer numbers being larger. Hence, they choose 4.63 as larger than 4.8 because it is longer and likewise 4.03 as larger than 4.3. This judgement is based on the successful strategy for whole numbers, where zeros contribute both to the length and size of a number.

Numerator focussed thinking: This is similar behaviour to String length thinking with one variation. Just as the whole number 38 is not changed by the addition of a preceding zero (038), the zeros after the decimal point are likewise disregarded. Hence, 6.3 and 6.03 would be seen as the same number; 6 wholes and 3 more parts. The disregarding of the size of the parts (first tenths and then hundredths in the above numbers) and concentration on the number of parts (here 3) has suggested the name for this type of thinker.

Reverse thinking: A student who has not heard the *th* in the place value names (or who has heard it but disregards it as it has no meaning for them) may believe that the decimal columns represent more whole numbers but written in the reverse order: i.e., (point) tens, hundreds, thousands etc. Evidence for this confusion was obtained in an interview with Val in Grade 5. When asked to read 0.163 from a card she replied "one hundred and sixty three....because when we do that in class we had a tens column and a hundreds column and a thousands column....I'm not sure if its just one hundred and sixty three or its 1 ten, 6 hundreds and 3 thousands".

Zero makes small thinking: This is a slight improvement on the Numerator focussed thinking above. The extra information these students have learnt is that a decimal with a zero or zeros in the first column(s) after the point is "small" and so they are able to make a correct decision on the comparison test for items like 4.08 and 4.5. Otherwise, they choose the longer decimal as larger. This is one thinking pattern behind the "zero rule" of Resnick et al (1989).

Right hand overflow thinking: This thinking pattern also results in "zero rule" behaviour of Resnick et al (1989). These students have created their own decimal version of column overflow. Whilst overflow to the left occurs in whole numbers, for example 120 is 12 tens, such students think 0.12 is twelve ten*ths* . In effect the number 12 is being squashed into the tenths column. Similarly, both 0.06 and 0.067 would be read as hundredths, first 6 then 67. Note that Right hand overflow thinkers cannot be reliably distinguished from Zero makes small thinkers by a decimal comparison test.

Shorter-is-larger Misconceptions

Denominator focussed thinking: These students have been identified by previous researchers (eg Resnick et al). They use the place value column names to decide on the size of the decimals. They know that one tenth is larger than one hundredth and they incorrectly generalise that any number of tenths is greater than any number of hundredths. While they may or may not have an image of the equivalent fraction, the consideration of the size of the parts (in isolation) has determined their choice and is reflected in the name of this group. As they demonstrate reasonable understanding of place value names and make sensible decisions with decimals of equal length, it would be easy for their wrong thinking to go undetected in the classroom.

Reciprocal thinking: These students have attempted to connect decimals with fractions but have difficulty as the decimal form does not explicitly indicate the denominator. Hence,

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when faced with the notation 0.3 they perceive that there are 3 parts rather than 10 equal parts of which 3 are chosen. For such thinkers 0.3 is larger than 0.4 as 1/3 is larger than 1/4, whether they actually write these symbols or not. Evidence for confusion between decimals and reciprocals was obtained in the interviews when Jack (a tertiary student) discussed how he had been thinking. He explained that just as "1 over 25 a fraction is a smaller number than 1 over 5". When he saw "oh point large number I immediately assumed in the same way that 1 over 25 is smaller ..thus that decimal was in fact smaller".

Negative thinking: These students are confusing notation for decimals with that for negative numbers. Amanda, a tertiary student, incorrectly compared 0.35 / 0.20 and then wrote: "I was thinking along a number line and considering decimal numbers to be equivalent to negative numbers. Therefore -20 was larger than -35". This leads to the same behaviour as Reciprocal thinkers on all the decimal comparison tests. Comparison items with one of the numbers being zero are required to separate them.

Apparent-expert Behaviour

Students who are able to correctly compare most pairs of decimals are grouped together here. Some of them are indeed experts, having a solid understanding of decimal notation. Others may not have a good understanding, but correctly follow rules to order decimals. For example, they may annexe zeros on the shorter decimal and then compare the resulting integers after the decimal point, or they may follow the procedure of comparing digits from left to right but have little understanding why this rule works. A written test cannot distinguish between these various forms of thinking, but it may be straightforward in an interview situation. In order to stress that students who do well may not have a good understanding of decimal notation, we use the term Task expert, rather than expert, to describe those who answer all types of questions correctly.

Item-by-item analysis of the decimal comparison test by Stacey and Steinle (1998) uncovered other groups of students who could generally compare decimals but who had very little appreciation of the place value basis of decimals.

Truncation thinking (2dp): Students (and adults) who think this way may use the contexts of money or length (m, cm) in order to make sense of decimal notation. Hence they focus on the first two decimal places and ignore the rest of decimal number. Rather than understand that they are actually making an approximation, sensible as it may be, they have no idea what these extra digits mean. So 15.348 may be seen as $15 and 34c or 15m and 34 cm; the 8 is some error or variation but they are unclear of what effect it has on the size of the number. Just as a sequence of house numbers can be written as 134, 135A, 135B, 137... it is understood that there is more than one house at 135, but ordering 135 and 135B may be difficult. Nancy, a university student, wrote about her uncertainty on some items of the decimal comparison test. Some items presented no difficulty (eg 4.08 / 4.7 became $4.08c / $4.70c) but "when the numbers are same (eg 4.4502 / 4.45) in the same spot I get very confused...Does the number get bigger or smaller with more numbers on the end?"

Rounding thinking (2dp): This is an improvement on truncation thinking as the size of the discarded digit is taken into account. So, 15.348 would be interpreted as analogous to $15 and 35c or 15m and 35 cm. Again, the additional digits after two places (or sometimes 1) have little or no meaning to such a student. While this is a sensible approach to quoting a result after a calculation is made, it demonstrates that the underlying decimal notation is not understood.

Students who think like this (Truncate/Round) may appear, in the classroom, to be experts. We have seen evidence of students who truncate to only one decimal place but truncation to two decimal places (probably deriving from money analogies) is more common.

Unclassified behaviour

Some students consistently use one interpretation of decimals, thereby being readily classified by a test using simple items such as decimal pair comparison. Other students may not have any of the above mentioned thinking patterns. Still others will have a mixture of several of these ideas, which they use inconsistently. Such students are labelled as Unclassified. There is some evidence (Moloney and Stacey, 1997) that students who are beginning to develop more sophisticated ideas about decimals will answer most inconsistently.

The description of patterns of interpreting decimal notation above is extensive, but not comprehensive. For example, there are further combinations of these patterns possible. In the rest of this paper, we examine which of the categories above can be easily identified in school populations.

Method, Data and Procedures

Development of the new test

In order to find the incidence of the misconceptions and thinking patterns described above, a new decimal comparison test consisting of 30 items was developed. As in previous tests, each item presents a pair of decimal numbers with the instruction to select the larger. Even with 30 items, the practical limit of such a test for children, some categories of decimals (e.g., those with final zeros, decimals with same integer part) are not adequately represented.

The decimal comparison items are in six groups of comparison items. The first two groups are used for the primary classification as shown in Table 1. In Group I test items, the five decimal comparisons have been selected so that the shorter decimal is the larger. For ease of reference, the larger decimal is listed first throughout this paper, although they are randomly presented in the test.

As shown in the table, students using one of the Longer-is-larger misconceptions (String length, Numerator focussed, Reverse, Right hand overflow or Zero makes small) will score poorly on this group of items, whereas those using one of the Shorter-is-larger misconceptions (Denominator focussed, Reciprocal or Negative) will score well. Apparent-expert students (Truncate, Round or Task expert) will also score well.

The five items in Group II have been selected so that the longer decimal is the larger, and the responses by the three primary classifications are listed in the table: only the students using one of the Shorter-is-larger misconceptions will have difficulty here. Low and High indicate performance within each group of items: Low indicates at most 1 correct while High indicates at most 1 incorrect. Hence, students were allowed to make one deviation from their expected responses within each group and still be classified. Students not following any of these patterns are unclassified. Hence, a medium score of 2 or 3 out of 5 in either of the two groups of items will result in a student being unclassified, as will the score of Low in both groups.

Table 1: *Definition of Primary Classification*

Primary classification	Group I 5 items eg 4.8 / 4.63	Group II 5 items eg 5.736 / 5.62
Longer-is-larger misconceptions	Low	High
Shorter-is-larger misconceptions	High	Low
Apparent-expert behaviour	High	High
Unclassified	Any Medium, or combination of Low and Low	

Once the primary classification has been made, the results of the next four groups of decimal comparisons are considered. For example, a student using String length thinking will have exhibited Longer-is-larger behaviour and would then (see Table 2)

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score Low in Group III, and High in Groups IV, V and VI. It can now be seen that it is not possible, using these six groups of decimal comparisons, to distinguish between some of the thinking patterns. For example, Right hand overflow and Zero makes small thinking cannot be separated; the same is true of Reciprocal and Negative thinking (this would need a comparison of a decimal with zero, an item that has not been included). As String length and Numerator focussed thinking present the same responses to the six groups of test items, an additional item (0.53 / 0.006) has been included. Ideally, there would be at least three such items in a group, in order that a student's way of thinking is not determined by their choice on a single item, but space limitations prevented this.

Table 2: *Definition of Refined Classification*

Refined Classification of thinking	Group III 4 items eg 4.7/4.08	Group IV 4 items 4.4502/4.45	Group V 3 items 0.4/0.3	Group VI 3 items 0.42/0.35
Longer-is-larger misconceptions:				
String length (*)	Low	High	High	High
Numerator focussed (*)	Low	High	High	High
Reverse	Low	High	High	Low
Right hand overflow	High	High	High	High
Zero makes small	High	High	High	High
Shorter-is-larger misconceptions:				
Denominator focussed	High	Low	High	High
Reciprocal	High	Low	Low	Low
Negative	High	Low	Low	Low
Apparent-expert behaviour:				
Truncate/Round (guess shorter)	High	Low	High	High
Truncate/Round (guess longer)	High	High	High	High
Expert (various understandings)	High	High	High	High

(*) These two ways of thinking are split using one additional item

Data collection and method

The 30 question test was given to 2517 students at 6 primary and 7 secondary schools across Melbourne during 1997. Table 3 shows the number of students tested at each year level. The items were presented in their groups, with some extra items at the beginning of the test. These 'warm-up' items were included because the results of our last data analysis (Stacey & Steinle, 1998) suggested that the earliest items were more likely to be answered inconsistently. The test was presented to all students on a single sheet with the larger decimal in each pair to be circled. Our previous investigation showed that this was a good arrangement to maximise the chances that students would respond in a consistent manner. Students were classified using the scheme shown in Tables 1 and 2.

Table 3: *Numbers of students tested at each year level.*

Year level	5	6	7	8	9	10	Total
Number of students	294	319	814	457	350	283	2517

Results and Discussion

Table 4 shows the resulting distribution of students across the whole sample. The three primary classifications account for about 85% of the students. It is surprising to see the identical numbers (13.1% each) within the two length-based misconceptions. We believe this is an accident of the age composition of the sample (see Table 3 and Figure 1). Most of the students in the Longer-is-larger misconception category (11.1% out of 13.1%) were able to be further classified, as were 9.5% out of 13.1% in the Shorter-is-larger misconception category, and 54.1% out of 58.2% in the Apparent-expert behaviour category. This is evidence that the identified thinking patterns explain most students' behaviour.

Table 4: *Results of Primary and Refined Classifications*

Primary and Refined Classifications	Number	Percentage
Longer-is-larger (including)	329	13.1%
String length	*48*	*1.9%*
Numerator focussed	*156*	*6.2%*
RH overflow / Zero makes small	*69*	*2.7%*
Reverse	*7*	*0.3%*
Shorter-is-larger (including)	329	13.1%
Denominator focussed	*72*	*2.9%*
Reciprocal / Negative	*167*	*6.6%*
Apparent-expert (including)	1464	58.2%
Task expert	*1269*	*50.4%*
Truncate /Round(shorter)	*94*	*3.7%*
Unclassified	395	15.7%
TOTAL	2517	100.1%

The refined classification shows that String length and Numerator focussed thinking dominate the Longer-is-larger misconception as suggested by earlier studies. An additional item in the test (0.53/0.006) allows the separation of the String length thinking and Numerator focussed thinking. This gives an estimate of one student using String length thinking to every three using Numerator Focussed thinking. The number of students showing Right hand overflow or Zero makes small thinking is about 3%. This is consistent with the proportion reported by Moloney and Stacey (1997). Identifying students demonstrating Reverse thinking has not been attempted before. The number of students in this category is rather small, and somewhat disappointing as the Group VI items were included specifically to detect this way of thinking. This thinking has been voiced in one of a small number of interviews with Grade 5/6 students and elsewhere. It is still possible that there are more students who think in this way (or with a variation of it) but the decimal comparison test format does not elicit this response. Just as interviews with task experts showed that the students may have other conflicting ideas, the procedures taught for ordering decimals may override students "reverse" ideas when they are faced with this task.

The results for the Shorter-is-larger misconception category are unexpected: the number of students using Reciprocal/Negative thinking is twice that of the Denominator focussed thinking. Earlier authors have assumed that Denominator focussed thinking was the main reason for Shorter-is-larger misconceptions. The possibility of Reciprocal or Negative thinking was not investigated previously because items where the decimal pairs are of equal length have not been included in earlier studies. This finding will alter the priorities for assistance to students in the Shorter-is-larger category.

Within the Apparent-expert behaviour category about 4% of the overall sample were classified as using Truncate/Round (shorter) thinking. These students have been identified by their incorrect choices to the Group IV items. The existence of this thinking was identified by Stacey and Steinle (1998) but without any idea of the number of students involved. Such thinking will be hard to detect in the classroom as the student will make sensible decisions on almost every occasion: only when the decimals involved start with the same digits (eg 0.56 and 0.561) will their understanding show up to be inadequate. Interviews with such students suggests their understanding of decimal notation is quite superficial. They are in fact using a system of two whole numbers separated by a dot, appreciating that the largest 2-digit number after the dot is 99. Because students using this thinking can only guess the answers to the four Group IV items, the number using such thinking may actually be about twice this size, the rest of the Truncate/Round thinkers happening to guess the correct (longer) answer and hence fall into the Task expert group. In a group of 40 tertiary graduate students (non-maths majors) who completed the comparison test, four students expressed difficulty on the Group IV items, suggesting that this way of thinking about decimals may be even more common amongst adults.

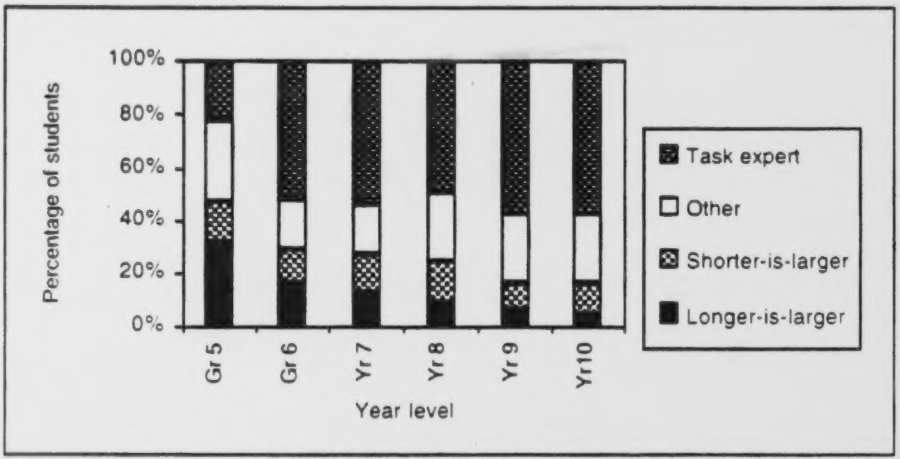

Figure 1: Distribution of classification by grade/year level

The distribution of the primary classifications within each grade/year level is presented graphically in Figure 1. The results confirm the age distribution reported by Moloney and Stacey (1997). The Longer-is-larger category decreases from Grade 5 to Year 10, the trend suggesting that it is unlikely to be common in adult life. The Shorter-is-larger category plateaus at about 10% which suggests that this general belief may continue into adulthood. The percentage of Task experts also plateaus to about 60% in Year 10, which suggests that there are many adults as well as students who have difficulty understanding decimal notation.

Table 5: *Average percentage of students by grade/year level and by classification*

	Longer-is-larger misconceptions			Shorter-is-larger misconceptions			Task expert		
	min%	**av%**	max%	min%	**av%**	max%	min%	**av%**	max%
Gr 5	14	**32**	52	7	**15**	30	9	**23**	30
Gr 6	7	**17**	48	0	**12**	37	0	**52**	82
Yr 7	2	**13**	21	2	**14**	28	31	**54**	85
Yr 8	1	**9**	23	12	**16**	18	30	**49**	70
Yr 9	5	**6**	7	6	**10**	14	54	**58**	63
Yr 10	4	**5**	10	8	**11**	14	52	**58**	69

Table 5 gives the average percentage of each grade/year level who are Longer-is-larger, Shorter-is-larger or Task experts. The table also gives the maximum and minimum percentage of students in a given category at any school. For example, the percentage of all the Grade 5 students in the sample choosing the shorter decimal as the larger is 15%, but at one school there were 30% and at another there were only 7%. All other schools fell between these extremes. Likewise, 52% of the Grade 6 sample were Task experts, but the results by school varied from 0% to 82%.

Conclusion

The purpose of this paper was to report on the incidence of the newly identified ways of thinking about decimal notation as revealed by the new test. The new test was not able to separate all types of thinking, because of logical and practical limitations. Ways of thinking that could not be split by the test (e.g., Right hand overflow and Zero makes small, Reciprocal and Negative) accounted for small total percentages, making their separation a low priority. One way of thinking which interview evidence had caused us to believe may be quite prevalent (Reverse thinking) was not prevalent in the results, either because it is indeed rare or because students bring different ideas to bear in the items that the test involves. In hindsight, it would have been better to omit the Group VI items (designed to separate the Reverse thinking) and replace them with more items such as 0.53/0.006 to allow a more accurate division of String length and Numerator focussed thinking. The group of students who have knowledge of only the first one or two decimal places was found to be at least 3.7% and probably twice this size. This is an important finding.

The variability of the results by school can in part be explained by socio-economic factors. However, there also seems to be clear evidence that certain misconceptions are learned from school instruction. Some of the peaks may be temporary effects, so that students may be more likely, for example, to show a shorter-is-larger misconception immediately after a unit on fractions. The number of Task experts in a class is likely to be temporarily high if testing takes place immediately after instruction on decimal comparison. Further analysis of our data will investigate questions such as this and identify effects at the class rather than the school level.

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Kindergarten Students' Progress in the Count Me In Too Project

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In 1997, 866 Kindergarten students, including 47 Aboriginals or Torres Strait Islanders, participated in the Count Me In Too Project. Their progress in early arithmetical strategies, forward number word sequences, and numeral identification was examined. Progress was compared with expected syllabus outcomes and it was found that the majority of the students met or exceeded those expectations. Consistent with earlier and smaller-scale studies, many students began the Kindergarten year with relatively high levels of knowledge and there was much diversity in levels.

In the state of New South Wales the primary mathematics curriculum which schools are expected to follow, is set out in the *Mathematics K-6* (1989) syllabus document. Included in this document are: aims, a statement of principles, information about assessment and evaluation, and mathematical content organised into the three strands of space, measurement and number. The document also outlines expected learning outcomes and includes teaching and learning units, which are grouped into three broad levels and which aim to support the realisation of the learning outcomes. Of particular interest to this report are those learning outcomes associated with number which are expected of students at the end of Kindergarten (their first year of school). These learning outcomes are more comprehensively outlined in a later document which deals with the three broad levels set out in the syllabus as well as *Early Stage 1* outcomes which "relate specifically to the expected achievement of students by the end of Kindergarten." (Board of Studies, 1997, p.2). Outcomes included which relate to number are:

sorts and describes objects in terms of their features such as size and shape;
recognises, describes, makes and continues simple number and spatial patterns;
recognises and compares the sizes of groups through a variety of strategies such as estimating, matching one-to-one, counting;
divides groups into parts of equal and unequal size through sharing, matching and counting;
represents numbers in a variety of forms, including the use of a calculator;
asks and responds to mathematical questions using drawing, making, describing, acting, guessing and checking, and retelling;

During the last decade the appropriateness of current mathematics curricula for young students has been questioned by several reseachers. Young-Loveridge (1989) conducted interview-based assessments of 81 beginning-school students in New Zealand and found that these students were being taught certain concepts, such as pattern recognition, even though they already knew them. Similarly, Aubrey (1993) reported concern about the UK curriculum where students in their first year of school "clearly enter school having acquired already much of this mathematical content" (p.39). Wright (1994) reports that generally school programs typical of the first year of school fail to take adequate account of the prior knowledge those students bring to school. Indeed the NSW syllabus itself supports the importance of challenging students "to achieve each task they undertake to the best of their ability" (NSW Department of Education, p.41) as does the national statement on mathematics when it claims that "(m)athematics learning is likely to be enhanced by challenge within a supportive framework" (Australian Education Council, p.20).

This paper focuses on the levels of knowledge of Kindergarten students towards the beginning and end of their first semester, and comparisons of these with expected syllabus outcomes. This leads to a consideration of the appropriateness of the NSW curriculum. The students were the Kindergarten cohort of participants in the Count Me In Too (CMIT) Project in NSW government schools. As well, the levels of knowledge of

47 Kindergarten students who were identified as Aboriginal or Torres Strait Islanders (ATSI) is considered at the beginning and end of the same period.

The Count Me In Too Project

Count Me In Too (CMIT) is a major school-based and systemic initiative of the NSW Department of Education and Training (DET) focusing on assessment and teaching in early years mathematics. Its basic goal is for teachers to better understand children's mathematical strategies and their development from less sophisticated to more sophisticated strategies. CMIT began in 1997 as a project subsequent to a project in 1996 (entitled Count Me In) which was piloted in four districts and involved at least three schools per district (Bobis, 1997). CMIT has been implemented in several schools in each of the 40 NSW school districts. Implementation of CMIT involves the district mathematics consultant working with all K-2 teachers in the school. CMIT does not take the form of a packaged program. Rather, it is a continually developing and evolving initiative. CMIT initially focused on the early number strand but this has been extended to include multiplication and division, and is currently being extended to include the early space strand. Development of Count Me In and Count Me In Too drew significantly on the theory and methods of the Maths Recovery Program (Wright, Stanger, Cowper & Dyson, 1996; Wright, Cowper, Stanger & Stewart, 1995). Key parts of CMIT adapted from Maths Recovery are the Learning Framework in Number (DOSE, 1997; Wright, 1997) and the Schedule for Early Number Assessment (see below).

Method

One aspect of CMIT in 1997 involved all participating schools administering initial and final assessments to selected students in Kindergarten, Year 1 or Year 2. The schools involved in Count Me In Too were selected by each school district. The criteria used in the selection of schools emphasised identifiable low socio-economic indicators, such as those used to identify schools on the Disadvantaged Schools Component of the National Equity Program. Assessment data was obtained on a total of 2288 students from 47 schools. This paper focuses on the assessment data from the Kindergarten cohort of this group — 866 students in all, whose ages ranged from 4.1 to 6.0 years. The 866 students included 47 students (or 5%) who were identified as Aboriginal or Torres Strait Islanders (ATSI). In selecting the ATSI students no attempt was made at using a stratified random sample or even a representative sample of disadvantaged schools. However, 2622 Kindergarten students were identified as ATSI out of 66 011 Kindergarten students in government schools in 1997 (i.e. 4%). Thus, the proportion of ATSI students involved in the 1997 Count Me In Too project was commensurate with that in all departmental Kindergarten classes. The ATSI students came from both country and metropolitan areas with the majority from the North of NSW. Assessment involved use of the Schedule for Early Number Assessment (SENA), which is individually administered and involves videotaping of student responses for subsequent analysis. SENA is a shortened version of the Maths Recovery assessment procedure (eg Wright, in press; Wright,1996; Wright et al., 1996) and includes the determination of the student's levels of achievement on various aspects of early number. This paper focuses on the levels of achievement on three aspects, viz Early Arithmetical Strategies, Forward Number Word Sequence (FNWS) and Numeral Identification. Initial assessments occurred around March 1997. During 1997, the Kindergarten teachers in the participating schools, revised their mathematics programs with the support of their district consultants. This involved teachers participating in school- and district-based professional meetings, and taking account of relevant aspects of CMIT, including the Learning Framework in Number and related video- and text-based material. Each of the three aspects which are the focus of this paper is explicated in tabular form via a progression of stages or levels of student achievement (See Tables 1-3 below).

Syllabus expectations in terms of these three aspects: Given that typical classroom practice in the first year of school is to focus on sorting and classifying activities, and to spend extensive amounts of time in studying numerals and number words up to "ten", then it seems resonable to claim that syllabus expectations in terms of the above three aspects are as follows: Early Arithmetical Strategies – Stage 1, can count perceived objects; FNWS – Level 3, facile with forward number words up to "ten"; Numeral Identification – Level 1, can identify numerals in the range "one" to "ten".

Table 1
Model for Development of Early Arithmetical Strategies

Stage 0: Emergent Counting. Cannot count visible items. The student either does not know the number words or cannot coordinate the number words with items.

Stage 1: Perceptual Counting. Can count perceived items but not those in concealed collections. This may involve seeing, hearing or feeling items.

Stage 2: Figurative Counting. Can count concealed items but counting typically includes what adults might regard as redundant activity. Thus, when presented with a collection partitioned into two parts (both screened), told how many in each part and asked how many counters in all, the student will count from "one" instead of counting on.

Stage 3: Initial Number Sequence. Student counts-on rather than counting from "one", to solve addition or missing addend tasks. The student may use a count-down-from strategy to solve removed items tasks (eg 17-3 as 16, 15, 14; — 14) or count-down-to strategies to solve missing subtrahend tasks (eg 17-14 as 16, 15, 14 ; — 3).

Stage 4: Facile Number Sequence. The student uses a range of non-count-by-one strategies. These strategies involve procedures other than counting by ones but may also involve some counting by ones. Thus in additive and subtractive situations, the student uses strategies such as compensation, using a known result, adding to ten, commutativity, subtraction as the inverse of addition.

Table 2
Model for the Construction of Forward Number Word Sequences (FNWSs)

Level 0: Emergent FNWS. The student cannot produce the FNWS from "one" to "ten".

Level 1: Initial FNWS up to "ten". The student can produce the FNWS from "one" to "ten". The student cannot produce the number word just after a given number word in the range "one" to "ten". Dropping back to "one" does not appear at this level. Students at Levels 1, 2 and 3 may be able to produce FNWSs beyond "ten".

Level 2: Intermediate FNWS up to "ten". The student can produce the FNWS from "one" to "ten". The student can produce the number word just after a given number word but drops back to "one" when doing so.

Level 3: Facile with FNWSs up to "ten". The student can produce the FNWS from "one" to "ten". The student can produce the number word just after a given number word in the range "one" to "ten" without dropping back. The student has difficulty producing the number word just after a given number word, for numbers beyond ten.

Level 4: Facile with FNWSs up to "thirty". The student can produce the FNWS from "one" to "thirty". The student can produce the number word just after a given number word in the range "one" to "thirty" without dropping back. Students at this level may be able to produce FNWSs beyond "thirty".

Level 5: Facile with FNWSs up to "one hundred". The student can produce FNWSs in the range "one" to "one hundred". The student can produce the number word just after a given number word in the range "one" to "one hundred" without dropping back. Students at this level may be able to produce FNWSs beyond "one hundred".

Results

Of the 866 Kindergarten students the following data was available: data on Early Arithmetical Strategies was available for 859 students; data on FNWS was available for 859 students; and data was available on Numeral Identification for 844 students. The data of 47 ATSI students who were part of the preceding larger sample, were also separately considered. Student progress in terms of each of the three models is examined. The results are reported in the form of summary tables. Table 4 shows the progress made by

Kindergarten students in terms of FNWS, Table 5 shows the progress made by Kindergarten students in terms of Numeral Identification and Table 6 depicts students' progress in terms of Early Arithmetical Strategies. Tables 7 to 9 show similar information for the 47 ATSI students. Each table depicts numbers and percentages of students assessed at a particular Level or Stage, at their initial interview around March, and at their final interview after at least ten weeks of teaching, typically early August. The results of an analysis of combined ATSI data have also been included.

Table 3
Model for the Development of Numeral Identification

Level 0: Emergent Numeral Identification.
Cannot identify some or all numerals in the range "1" to "10".
Level 1: Numerals to "10"
Can identify numerals in the range "1" to "10".
Level 2: Numerals to "20"
Can identify numerals in the range "1" to "20".
Level 3: Numerals to "100"
Can identify one and two digit numerals.
Level 4: Numerals to "1000"
Can identify one, two and three digit numerals.

Table 4.
Numbers of Kindergarten Students at Given FNWS Levels on Initial and Final Assessments.

	Number of students at a given level						
	0	**1**	**2**	**3**	**4**	**5**	**Total**
Initial assessment	237	276	93	156	74	23	859
	(28%)	(32%)	(10%)	(18%)	(9%)	(3%)	(100%)
Final assessment	50	141	112	211	196	149	859
	(6%)	(16%)	(13%)	(25%)	(23%)	(17%)	(100%)

Table 5.
Numbers of Kindergarten Students at Given Numeral Identification Levels on Initial and Final Assessments.

	Number of students at a given level					
	0	**1**	**2**	**3**	**4**	**Total**
Initial assessment	410	283	69	51	12	844
	(49%)	(34%)	(8%)	(7%)	(2%)	(100%)
Final assessment	130	295	129	250	40	844
	(15%)	(35%)	(15%)	(30%)	(5%)	(100%)

Points Arising From The Tables Involving The Kindergarten Cohort

FNWS initial: Table 4 shows that a large number of students (28% of the total sample) were Level 0 at the time of their initial interview. These students were unable to produce the FNWS from "one" to "ten". Twenty nine percent of the students were already at Level 3 or higher. In their Kindergarten year these students had met or exceeded syllabus expectations by March.

FNWS final: At the time of their final assessment 25% of students had reached Level 3 and 40% had reached Level 4 or Level 5. However, 35% of students, including 6% who were still at Level 0, remained below Level 3. They were not facile with the FNWS up to "ten" and, thus had not yet met the syllabus expectations for the end of Kindergarten.

Numeral Identification initial: Table 5 shows that at the time of the initial interview 49% of students were at Level 0. Thirty four percent of students were already at Level 1, that is able to identify numerals in the range one to ten, ie they had reached expected syllabus outcomes. Seventeen percent of students were initially at Level 2 or higher and had exceeded syllabus expectations.

Table 6.
Numbers of Kindergarten Students at Given Stages of Early Arithmetical Strategies on Initial and Final Assessments.

	Number of students at a given Stage					
	0	**1**	**2**	**3**	**4**	**Total**
Initial assessment	369	388	69	27	6	859
	(43%)	(45%)	(8%)	(3%)	(1%)	(100%)
Final assessment	101	416	187	122	33	859
	(12%)	(48%)	(22%)	(14%)	(4%)	(100%)

Numeral Identification final: At the time of their final assessment 35% of students had reached Level 1 and 50% had reached level 2 or higher. However, 15% of students remained at Level 0 at their final assessment. These students had not yet attained expected syllabus outcomes for the end of Kindergarten.

Early Arithmetical Strategies initial: Table 3 shows that at the time of the initial interview a large number of students (43% of the total sample) were at Stage 0, ie the Stage of Emergent Counting. The remaining 57% were already at Stage 1 or higher, thus meeting or exceeding syllabus expectations for Kindergarten students.

Early Arithmetical Strategies final: At their final interview 48% of students had reached Stage 1 and thus had met syllabus expectations. Forty percent of students reached Stage 2 or higher. However, 12% of students had not yet progressed beyond the Stage of Emergent Counting at the time of their final interview.

Progress of students identified as Aboriginal or Torres Strait Islanders

Table 7.
Numbers of ATSI Kindergarten Students at Given FNWS Levels on Initial and Final Assessments.

	Number of ATSI students at a given level						
	0	**1**	**2**	**3**	**4**	**5**	**Total**
Initial assessment	7	21	3	13	3	0	47
	(15%)	(45%)	(6%)	(28%)	(6%)		(100%)
Final assessment	0	9	4	10	14	10	47
		(19%)	(9%)	(21%)	(30%)	(21%)	(100%)

Table 8.
Numbers of ATSI Kindergarten Students at Given Numeral Identification Levels on Initial and Final Assessments.

	Number of ATSI students at a given level					
	0	**1**	**2**	**3**	**4**	**Total**
Initial assessment	29	16	2	0	0	47
	(62%)	(34%)	(4%)			(100%)
Final assessment	8	19	4	16	0	47
	(17%)	(40%)	(9%)	(34%)		(100%)

Table 9.
Numbers of ATSI Kindergarten Students at Given Stages of Early Arithmetical Strategies on Initial and Final Assessments.

	Number of ATSI students at a given Stage					
	0	**1**	**2**	**3**	**4**	**Total**
Initial assessment	24	22	1	0	0	47
	(51%)	(47%)	(1%)			(100%)
Final assessment	6	33	6	2	0	47
	(13%)	(70%)	(13%)	(4%)		(100%)

FNWS initial: Table 7 shows that only 15% of students were Level 0 at the time of their initial interview. These students were unable to produce the FNWS from "one" to "ten". Thirty four percent of the students were already at Level 3 or higher. In their Kindergarten year these students had met or exceeded syllabus expectations by March.

FNWS final: At the time of their final assessment 21% of students had reached Level 3 and 51% had reached Level 4 or Level 5. However, 28% of students remained below Level 3. They were not facile with the FNWS up to "ten" and, thus had not yet met the syllabus expectations for the end of Kindergarten.

Numeral Identification initial: Table 8 shows that at the time of the initial interview 62% of students were at Level 0. Thirty four percent of students were already at Level 1, that is able to identify numerals in the range one to ten, ie they had reached expected syllabus outcomes. Four percent of students were initially at Level 2 and had exceeded syllabus expectations.

Numeral Identification final: At the time of their final assessment 40% of students had reached Level 1 and 43% had reached level 2 or higher. However, 17% of students remained at Level 0 at their final assessment. These students had not yet attained expected syllabus outcomes for the end of Kindergarten.

Early Arithmetical Strategies initial: Table 9 shows that at the time of the initial interview a large number of students (51%) were at Stage 0, ie the Stage of Emergent Counting. Forty seven percent were already at Stage 1, thus meeting syllabus expectations for Kindergarten students.

Early Arithmetical Strategies final: At their final interview 70% of students had reached Stage 1. Seventeen percent of students reached Stage 2 or higher, thus exceeding syllabus expectations. However, 13% of students had not yet progressed beyond the Stage of Emergent Counting at the time of their final interview.

In addition to the analysis of ATSI data appearing in Tables 7-9, a simple summary analysis of ATSI data not appearing in those tables was undertaken. This data dealt with each student's performance in terms of all three aspects, ie FNWS, Numeral Identification and Early Arithmetical Strategies. The analysis examined the students' performance in a combined way on all the three aspects. The expected performance was defined as Level 3 on FNWS, Level 1 on Numeral Identification and Stage 1 on Early Arithmetical Strategies. Thus, a student identified as Level 2 on FNWS but Level 1 on Numeral Identification and Stage 1 on Early Arithmetical Strategies, was classified as having not yet achieved the syllabus expectations in numeration. A combined performance beyond the expectations of the syllabus would, for example, be attributed to a student assessed as Level 3 on FNWS, Level 2 on Numeral Identification and Stage 2 on Early Arithmetical Strategies. Defined in this way, the combined results for ATSI students involved in the project were as follows:

At the initial assessment:

- 11% were performing beyond the expectation of the Kindergarten syllabus, ie had numeral knowledge beyond "ten".
- 19% were meeting the expected performance, ie were perceptual counters with knowledge of numerals and number words up to "ten".
- had not yet met the syllabus expectations in numeration.

At the final assessment:

- 57% were achieving beyond the expectation of the Kindergarten syllabus, ie had numeral knowledge beyond "ten".
- 9% achieved the expected performance in all three aspects, ie were perceptual counters with knowledge of numerals and number words up to "ten".
- 34% had not yet met the syllabus expectations in numeration, ie demonstrated knowledge of numerals and number words up to "ten".

Discussion

The results of the 47 ATSI students will not be discussed specifically, since they were part of the larger Kindergarten sample and are, thus automatically included in the discussion. The results indicate that beginning-school students bring with them a diverse range of mathematical knowledge. For example, at the initial assessment 28% of students were unable to produce the FNWS from "one" to "ten", whereas 3% of students were facile with the FNWS up to "one hundred". Similarly, whilst nearly half of the students were unable to identify all or some numerals in the range" one" to "ten", 7% of the total sample of students was able to identify one and two digit numerals. However, despite this diversity, progress was made by the majority of students and syllabus expectations were not only reached but exceeded by many of these students coming from disadvantaged backgrounds.

At the time of their final assessment 25% of the Kindergarten cohort had met syllabus expectations in terms of FNWS, 35% of the cohort had met syllabus expectations in terms of Numeral Identification and 48% of students had met syllabus requirements in terms of Early Arithmetical Strategies. Forty percent of students exceeded syllabus expectations in terms of FNWS, approximately half of the students exceeded syllabus expectations in terms of Numeral Identification, and 40% of the students were at Stage 2 or higher at their final interview, thus exceeding syllabus expectations in terms of Early Arithmetical Strategies. Progress on such a large scale across such a geographically diverse range of 47 schools suggests that CMIT is a viable program for advancing many beginning-school students beyond expected syllabus outcomes. However, concern still remains for those students who had not met syllabus outcomes at the time of the final assessment.

At their final assessment 35% of Kindergarten students were not facile with the FNWS to "ten". Similarly, 15% of students were unable to identify all or some numerals in the range "one" to "ten". These students had not met syllabus requirements at the time of the final assessment. This was also the case for 12% of students who were unable to count visible items and thus remained at a stage of Emergent Counting. Other factors such as teaching style, learning environment or specific student attributes may have contributed to these students' apparent lack of success.

Finally, from the results it appears that, early in their first year of school a significant number of Kindergarten students were already able to complete the tasks, that were expected end-of-year syllabus outcomes. Thus 29% of students had already met or exceeded syllabus expectations in terms of FNWS, 51% of students had already met or exceeded syllabus expectations in terms of Numeral Identification and 57% of students had already met or exceeded syllabus expectations in terms of Early Arithmetical Strategies at the time of their initial assessment. This suggests that many students are under-challenged in their first year of school. This finding is consistent with earlier research (see Aubrey, 1993; Wright, 1994 ; Young-Loveridge, 1989).

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Task Context and Applications at the Senior Secondary Level

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A qualitative study was conducted to investigate the impact of prior knowledge of task context on students' approaches to applications tasks and the effect of engagement with context on performance. Prior knowledge mainly enhanced students' understanding of the task having most impact when students constructed a mental picture of the situation. Moderate to high engagement with a task context was not often associated with poor performance. Poor performance was more likely to be associated with no to low engagement. High engagement was not a necessary condition for success as the degree of engagement necessary for success appeared to be task specific.

Introduction

Recent syllabus documents in Queensland (eg, BSSSS, 1992) place an emphasis on applying mathematics to familiar and unfamiliar situations in a variety of contexts. Clarke (1994) warns, however, that instead of the use of contexts providing an emancipatory experience for students in the upper secondary school, for some it may be the source of a further level of difficulty and confusion. Amit (1994) echoes this warning: "as assessments become more complex and more connected to real-world tasks there is a greater [chance] that the underlying assumptions and points of view may not apply equally to all students" (p. 15). Research on the confounding or facilitating effects of context on performance are needed to inform the professional judgement of teachers in framing applications tasks, evaluating student responses, and providing appropriate learning experiences for students.

Selected Literature Review

Not all data in the presentation of an applications problem will have the same strength in cuing information, prior experiences, and processes necessary for its solution. The salience of specific cues for a particular problem solver will contribute to how they perform on a task (Kaplan & Simon, 1990). Cue salience interacts with the problem solver's domain specific prior knowledge and their knowledge of global strategies. Prior knowledge can have a beneficial effect constraining the search space for a problem representation (Helme, 1995) or the reverse effect leading to search in the wrong part of the problem space (Kaplan & Simon, 1990). Aberrant readings of a verbal task resulting from irrelevant prior knowledge interfering with the problem solver's reading of a task can also result in search in the wrong problem space. In applications tasks cue salience and its interaction with prior knowledge are of particular importance. Furthermore, problem solvers are unlikely to engage with the context of a task with which they are unfamiliar to the same degree as, or in a similar manner to, a person who has prior experience of the situation (Masingila, Davidenko & Prus-Wisniowska, 1996).

This paper reports part of a much broader study which investigates the effects of context on students' approaches to, and performance on, applications tasks in Mathematics B, the main pre-tertiary mathematics subject in Queensland. Specifically, the following questions are addressed here:

1. What effect does prior knowledge of the task context have on students' approaches to applications tasks?
2. What effect does engagement with context have on performance on applications tasks?

The effects of prior knowledge of the task context will be examined through an investigation of how students "exploit the texture of their experience as a resource in their mathematical activity" (Noss & Hoyles, 1996, p. 44). For adolescents in high school, this experience may be vicarious through knowledge developed in other academic subject areas or their general encyclopaedic knowledge of the world, or, truly experiential developing from their personal experiences outside school or in more practical school subjects. The terms *academic knowledge*, *encyclopaedic knowledge* and *episodic knowledge*, respectively, will be used in the following to distinguish between these sources of prior knowledge. An insight is sought into the interplay between the lived-in experience of students and their mathematical activity in attempting to solve these tasks.

The Study

The Participants

The study was conducted in two public high schools in a large provincial city. Data collection occurred over a six month period during which contact with both schools was continuous. Forty-three Year 11 and Year 12 students participated in the study.

Multi-Site, Multi-Case Design

The use of one site for a study such as this would allow the researcher "to describe, understand and explain what [had] happened in a single, bounded context" (Miles & Huberman, 1994, p.172). The purpose was to glean "a well-grounded sense of the local reality". Within that one site multiple individual case histories (eg, of students and teachers) could be studied and integrated to form a picture of that local reality. However, a multi-site design increases the applicability of the study's findings to other similar settings allowing idiosyncratic nuances of a particular site to be identified as well as determining what is of a more general nature. A more pressing reason for choosing a multi-site, multi-case design was to deepen understanding and explanation of a very complex setting (Miles & Huberman, 1994).

Development of Applications Tasks

In all, 18 applications tasks were developed. Illustrative examples are included in an appendix. These tasks differed in terms of familiarity, complexity and degree of contextualisation. An *unfamiliar application* was taken to mean an unrehearsed mathematical application to a life-related situation where the context (the situation) was unfamiliar to the students in the sense that they had not mathematized a similar situation previously but the mathematical techniques needed to solve the problem were not unfamiliar (BSSSS, 1994). The *complexity* of the task "depends on how obvious the choice of techniques is, the number of techniques required, and the amount of guidance given to students" (p. 1). The term *degree of contextualisation* is meant to convey the range of embeddedness that exists between the mathematics that can be used to model a situation and the description of the situation.

Administration of the Tasks

Students were videotaped as they attempted to solve a task individually. The role of the researcher during the task solving session was to be as unobtrusive as possible unless the student was experiencing grave difficulties accessing the problem when students were invited to ask clarifying questions and offered prompts to enable them to start in order to attempt to determine which aspects of the task were preventing them from accessing it. All interactions were recorded.

For all phases of the data collection except the second, students were given at least one task to complete and if time permitted, a second task was completed. During the second phase of data collection six students were chosen by the researcher, in consultation with the Head of Department, to attempt a series of four unfamiliar applications tasks that were of differing complexity and degree of contextualisation (namely, the Ecosystem Problem, Road Construction Problem, Road Accident Problem and Painting Problem). It turned out that one of the tasks (the Painting Problem) was similar to a task students had seen previously (the Picture Theatre Problem, Brodie & Swift, 1993, p. 252). It was decided not to discard the task but to use it to develop tasks that were parallel in mathematical content but embedded in different contexts to investigate the effect of context in these circumstances. To avoid presentation bias students attempted these tasks in different order. The basis for selecting particular students was their facility with mathematical techniques on school assessment, their success with applications tasks in the school assessment programme, and their degree of ease with the situational context in which the tasks were attempted as demonstrated in their first task solving session.

The Student Interview

Semi-structured stimulated recall interviews were conducted by the researcher immediately following the completion of the tasks. The interviews were recorded using videotape with audio-cassette tape as a backup. The students reviewed the videotape of their task solving session in conjunction with the script of the task during the interview except for 3 instances where technical difficulties prevented this when the script alone was used as a stimulus. Students were asked to draw diagrams where appropriate during the interview to illustrate their understanding of the context and problem goal as diagrams provide insight into students' understanding of the structure of a problem by revealing their understanding of the relationships between the various components of the problem (Shigematsu & Sowder, 1994). Nunekawa (1994) and Diezmann (1995) recommend monitoring the dynamic use of diagrams throughout the solution process to see the changes in relationship between students' perceived structure of the problem and the actual structure. The use of the video-taped task solving sessions as a visual stimulus throughout the interview allowed both the interviewer and the interviewee to track the student's developing understanding of the task structure by discussing their changing perception of the task as observed in their changing use of diagrams throughout the review of the task solving session.

Analysis

All interviews and task solving sessions where there were interactions between the student and the researcher were transcribed. The interviews were analysed using the qualitative data analysis software QSR NUD.IST (QSR, 1997). NUD.IST facilitates "grounded theory" construction (Strauss, 1987) which attempts to capture and interrogate the meanings emerging from data. This is achieved by constructing and exploring new categories and themes as they arise from the data then refining these through a "process of progressive category clarification and definition" (Tesch, 1990, p. 86). According to Tesch, the main theory building functions of NUD.IST are the location of places where categories intersect and "the corresponding search for counter-evidence" (p.165).

Results

Prior Knowledge of Task Context

Disconnections Between Students' Experience and Task Context: Unfortunately for students, there are often many disconnections between their lived-in world and the contextual situations described in classroom tasks. This is illustrated by the case of A56 whose claim that he knew "a fair bit" about mechanical things was substantiated by his description of what was happening in the Shaft Problem.

> A56(Task7): You have just got a shaft that has been worn down.
>
> I: Uhmm.
>
> A56(Task7): *...through spinning.*
>
> I: So you assume it was spinning to wear down?
>
> A56(Task7): Oh, yes.
>
> I: That would be more than likely?
>
> A56(Task7): *A cylinder, yes.* So it's changing the diameter of the shaft.

He was in the realms of his lived-in experience, his episodic knowledge, so he was quite willing to garnish his description with elaborations from his personal experience. He was much less confident in his description of the mathematics of the situation.

> I: In terms of maths, what sort of equation is it?
>
> A56(Task7): The derivative.
>
> A56(Task7): *Of something, yeah.*
>
> A56(Task7): And so I'd have to take the integral of it.
>
> I: All right. There you are. Get going.
>
> A56(Task7): *Awh, god. That's not one of the rules is it? Because if it is I wouldn't have a clue.*

His mathematical knowledge was something he had acquired at school. It was disconnected from his experience out of school even when the context of a task was something so familiar to him. This may be explained, however, by the fact that in a written representation many of the situational aspects of a problem are lost unless they can be structured in a way to arouse sense-making connotations, which means, according to Nunes, Schliemann, and Carraher (1993), not losing the referents in a situation that make it meaningful in the everyday world.

Sense-making Through Making Connections: During discussions with the interviewer the student was able to make connections between the mathematical meanings of the written task and the everyday meanings of the familiar situation as shown in the next exchange when he encountered the cube root of a negative number. When pressed for possible reasons for this number being negative his episodic knowledge of the context came to the fore.

> I: But the negative couldn't have been something like the amount of wear? It couldn't have been some indication of that?
>
> I: Now if it's negative, it's just showing you that the wear is decreasing, the rate [of wear] is decreasing. You don't think it was that situation?
>
> A56(Task 7): *Oh, yeah, because it is, although the wear would increase as time went by.*
>
> I: The actual amount of wear would increase as time went by?
>
> A56(Task7): Well, using a cylinder, a shaft, using the words instead of the numbers, because *once you get a little bit of gap it starts flogging in.*
>
> I: Yes, you get a lot more tool vibration.
>
> A56(Task7): *The tool vibrates more so you are going to get more wear.*
>
> I: Yes, so you would expect the wear to be increasing?
>
> A56(Task7): *Yeah.*

> I: The actual rate at which it is happening is increasing so another thing to explain the negative would be the diameter itself saying you are taking away from the diameter and the diameter is decreasing. That's the only other thing.

Looking at his mathematical activities through the meanings of his experiential knowledge is not something he usually tried to do. He had a tendency to work on the two separately and did not appear to realise that if he could integrate the contextual and mathematical meanings, the situational context could guide his solution throughout by helping him to make sense of what he was doing mathematically.

> A56(Task 7): Well, I go through the numbers and work out...get an answer first off, then look at the actual situation and if it won't suit the situation then, then I'm totally wrong.

In the remainder of this section, students' ability to establish some connection between the meanings implicit in the mathematics of the written tasks in the study and the meanings derived from their prior knowledge of the situational context, as just illustrated, will be investigated.

Types of Prior Knowledge Displayed: A case-level meta-matrix (Miles & Huberman, p. 178) of the data from the first round of collection (Year 12 data from School A) was used to categorise student responses to questions regarding what prior knowledge, if any, they had of the task context. The affirmative responses were categorised as academic knowledge, encyclopaedic knowledge and episodic knowledge by the researcher. The type of prior knowledge that students mentioned that they could have drawn on was mainly academic from subjects such as Biology and Multi-Strand Science or general encyclopaedic knowledge. There was only the one example (just described) where the student's knowledge was categorised as episodic. Of the ten instances of prior knowledge displayed, there were only three cases where it had an impact on the student's solution. These all involved an enhancement of the student's understanding of the task.

In the second round of data collection it was decided to vary the task format and tailor tasks more towards personal background experiences of students. Three of the tasks (Ecosystem Problem, Painting Problem and Road Accident Problem) drew on more general contexts with the potential to involve academic and encyclopaedic knowledge whereas the remaining tasks were expected to activate the episodic knowledge of students. The Road Construction Problem was chosen on the basis that it was believed to have potential to engage students' interest because road construction was something they would see on a daily basis as, at the time of the study, major road works were underway near the school. The remaining tasks had a sports context (rugby league football and ice hockey) as the researcher knew that several of the students involved in the study played or followed rugby league football or hockey and these sports were prominent in the school community. As the researcher was not as familiar with the backgrounds of the students at School B, she did not set tasks that she expected would necessarily tap into students' episodic knowledge at that site.

A case-ordered meta-matrix was used to analyse the student responses for the remaining rounds of data collection. These were then categorised as above. The type of prior knowledge of the context reported for the Year 11 sample from School A was as anticipated. The results from students who attempted multiple tasks confirmed that this was a task specific rather than student specific effect. Just what effect, if any, these different sorts of prior knowledge had on student solutions or student interest needs a closer examination, however.

There were three cases where student's prior knowledge interfered with their development of a solution, however, in two of these cases the students involved were able to overcome this. In the other case the student's academic and encyclopaedic prior knowledge interfered producing an aberrant reading of the task. There were a further 14 cases where the prior knowledge reported or displayed appeared to be redundant as it was not used at all. Only two of these were classified as episodic with the remainder evenly divided between academic and encyclopaedic. The remaining cases where student's prior knowledge had a positive impact on their solution attempt will be examined further.

Use of episodic knowledge: Only seven of the thirteen students who attempted the Road Construction Problem reported using their episodic knowledge in any way. Three students even denied that the situation of a road being built was something they knew about. When used, episodic prior knowledge had the effect of enhancing a student's understanding of the problem situation having the most impact when students used it to develop a mental picture of the situation.

With the parallel tasks to the Picture Theatre Problem, four students dismissed the context once they realised it was a parallel problem. Three other students who attempted the Ice Hockey Problem all played field hockey and reported using their prior knowledge to understand the problem situation and construct their representation of it as they had visualised a field and an ice hockey player skating towards a goal. A72 also reported being able to use her visualising skills throughout the problem to check that she was on track. The remaining student attempted the Rugby Goal Problem which he tried to solve using his tacit knowledge of the game.

> A66(TASK13): ...See, you want a plain angle of 45 degrees. You come back - sort of margin of error, sort of margin for error like. If you want to, if you want to go to the centre of that...[meaning the crossbar].
>
> A66(TASK13): You want to go for the centre out...and if you want that to be 45 degrees and that is 13 that will have to be 13 as well.

Only two students from School B reported any prior episodic knowledge of task context. B4's father was in property and she had seen preventative measures such as fences to stop the erosion of banks at the side of roadways. This only helped to make her more comfortable with her understanding of the context in the Road Construction Problem. B7, however, chose a periodic function as her model in the Tide Problem and used her episodic knowledge of tides to reassure herself that this was a sensible model.

Use of academic knowledge: Although five of the nine students who attempted the Ecosystem Problem and three of the seven students who attempted the Road Accident Problem reported having prior knowledge from academic subjects that would have assisted them in understanding the context of these problems, no student felt this assisted them with their solution processes greatly beyond this except A85.

> A67(TASK12): I understood what it was about - not how to do it.

A85 considered the Ecosystem Problem to be much more practical than the Road Construction Problem. This was in direct contrast to most other students who viewed the Ecosystem Problem as little more than an academic exercise.

> A85(TASK9): I liked this one actually because it was, umm, more practical to...I could understand it [more easily] than the other one - you know, the lane, the one with the lane. I couldn't visualise that one but I could [this one].
>
> A85(TASK9): Awh...ummm...I found it [the context] very [useful]. I liked it. Yeah, that was very easy for me to understand and it was a...umm...like a realistic problem for me.

The possible reason for this was her affinity with the context.

> A85(TASK9): I think I know in my mind I kept visualising fields of grassland and bushland and woodland.
>
> A85(TASK9): Yeah, I can relate to it. That's what I am trying to say.

When asked if it would have made a difference to her if the context had been jelly beans rather than ecotypes she replied:

> A85(TASK9): Umm, probably, because...umm...I wouldn't have been able to understand it with jelly beans because this I can sort of grasp the realness of it.

Her affinity with the context of this problem appears to come from her prior academic knowledge of the situation developed through her study of Geography together with her ability to transform the change diagram into something so concretely visual in her mind. With the Road Construction Problem she could not visualise the situation at all but this was influenced by an aberrant reading of the context initially. The students who said they did not have an academic knowledge of the context of these tasks did not feel disadvantaged, however. This was because they felt the context was easily understandable anyway, or they were dismissive of the context.

Use of encyclopaedic knowledge: Encyclopaedic knowledge of the task context mainly reassured students about their understanding of the problem. It allowed them to speculate on possible reasons for the events described in the tasks happening and whether or not they thought the scenarios were feasible. There were, however, three cases where encyclopaedic knowledge was used to choose between two mathematical options, eg, B1 used her encyclopaedic knowledge to choose between two possibilities in the Rain Problem.

> B1(TASK16): Well, you are going to...actually there is going to be two possibilities and *it is going to have to be negative because of the drought.*

In another two cases, encyclopaedic prior knowledge was used to check interim or final results. A85, for example, was able to judge that her final speed for the car which ran off the road in the Road Accident Problem was unreasonable.

> I: So, now you gave a little giggle at this stage? Why was that?
> A85(TASK12): The answer was ridiculous.
> A85(TASK12): Yes, very unrealistic unless you had a rocket.

Summary: Students in the study displayed prior knowledge of the task context which was of an academic, encyclopaedic or episodic nature. The types of prior knowledge reported appear to be related to the nature of the task and were not student specific. The effects of this prior knowledge on the development of a task solution varied. In a few cases it interfered with the student's development of a solution, in some cases it appeared redundant, not being used at all, but in the majority of cases it had a positive impact. In most cases prior knowledge enhanced students' understanding of the task having the most impact when it enabled students to construct a mental picture of the situation. There were a few cases where prior knowledge was used to enhance decision making enabling students to select an appropriate mathematical model or choose between two mathematical options. In other cases, prior knowledge was used to check progress or interim or final results.

Engagement with Context

Qualitative data from all cases were scaled by the researcher using the two dimensions degree of engagement in task context and quality of performance. The task videotapes and student reviews of these during the post-task interviews were examined to scale the cases along a continuum from no to high engagement. The task scripts were then used to scale the task solutions along a continuum from being unable to make a start or totally misinterpreting the problem to a totally correct solution with no conceptual or mechanistic errors. These two dimensions were then used as axes for scatterplots (Miles & Huberman, 1994, p. 197) where cases were positioned in the space formed by the respective axes so that similarities and contrasts among cases could be seen. In those cases where the researcher assisted the student two values have been shown with an arrow indicating the direction of the change from the point in the solution when the researcher intervened.

Figure 1 shows the scatterplots for the Year 12 and Year 11 samples, respectively, at School A. Few students were successful in the Year 12 sample and there was little to no engagement with the task context. This is in stark contrast to the Year 11 pattern with approximately half the cases showing at least partial success and some to high engagement with the task context. The clustering of cases by task foreshadowed the possibility that there could be task specific effects so the cases for particular tasks were plotted on separate scatterplots (Figure 2).

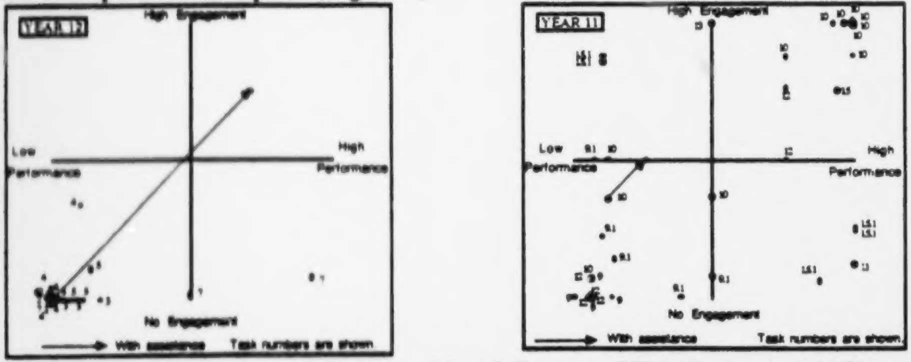

Figure 1. Scatterplots of Relationship of Degree of Engagement with Context and Performance on All Tasks at School A by Year Level

From Figure 2 it would appear that to be successful on the Road Construction Problem a high degree of engagement with task context is necessary. In contrast, the Ecosystem Problem does not appear to require a high degree of engagement with the context to be successful although the students in the study had only limited success with this problem. The results for the parallels with the Picture Theatre Problem appear to be contradictory. A confounding factor with this problem was the fact that some of the students used their knowledge of the solution to the parallel problem rather than developing their solution from the context. These students were quite successful but only one of them reported

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relying significantly on the context. If these students had not seen the parallel problem they would have had to engage with the context a lot more to construct a suitable representation for the problem. The other two students made little progress as they both made a similar misinterpretation of the Ice Hockey Problem believing that the skater was restricted to skating towards the goal only in the two metre width of the goal mouth. Success with the Road Accident Problem appears to be related to the level of engagement with task context but the sample of cases is small so this would need further confirmation. The more successful students on this task, B5 and A72, displayed a much higher level of engagement with the task context than the other students.

Figure 3 shows a scatterplot of the data from School B. The level of engagement with task context varies for the successful students with the particular task ranging from none to high. The unsuccessful students, however, were all unable to engage, without assistance, to a significant extent with the context of tasks which other students had found it necessary to do so to be successful.

A cross-site scatterplot is shown in Figure 4 for all the tasks. It would appear that moderate to high engagement with task context is not often associated with poor performance. Poor performance on tasks is more likely to be associated with no to low engagement with the task context. High engagement with the task context is not a necessary condition for success as it appears to be dependent on the nature of the task as shown in Figure 2.

Discussion of Results and Implications

The results from this study indicate that prior knowledge of the situational context of an applications task can have a positive effect on a student's understanding of a task especially for those students who have well developed visualisation skills. However, the effects of prior knowledge are not always positive with it sometimes producing no effect or actually interfering with a student's understanding of a task. The findings of this study are therefore in agreement with Clarke (1994) and Amit (1994). Episodic prior knowledge is more likely to have a positive effect than academic or encyclopaedic knowledge. Teachers also need to realise that not only will the effects of prior knowledge differ from student to student because of differing backgrounds but also the fact that particular students have had the same background experiences does not mean that memories of those experiences will be activated for all students when relevant or have similar impact on the task solution when they are activated.

Many students in the study were unable to engage with the context of an

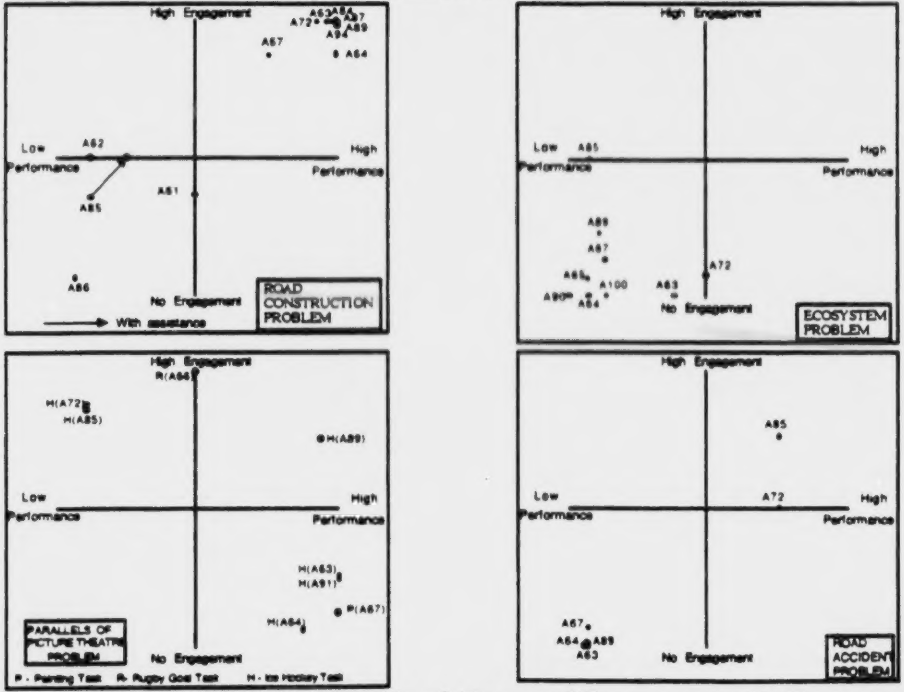

Figure 2 Scatterplot of Relationship of Degree of Engagement with Context and Performance on Specific Tasks at School A(Year 11)

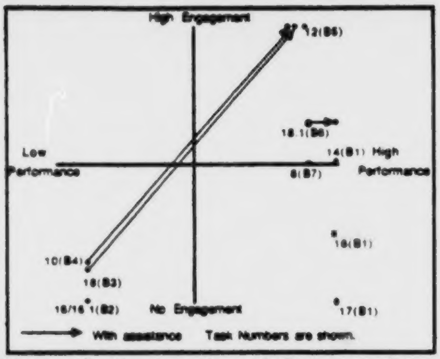

Figure 3. *Scatterplot of Relationship of Degree of Engagement with Context and Performance on Specific Tasks at School B*

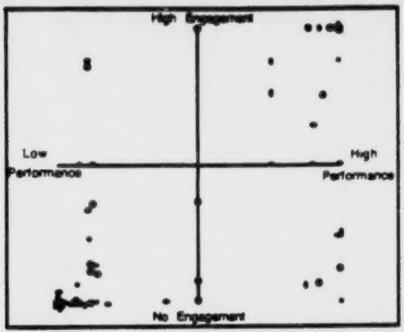

Figure 4. *Scatterplot of Relationship of Degree of Engagement with Context and Performance on All Tasks Across Sites*

applications task to any significant degree particularly when tasks were used that were similar to those used in the classroom and school assessment program (eg, School A Year 12). This would not be problematic if tasks are set which avoid the necessity for such engagement for success, however, if the rationale of syllabus documents is to be taken seriously tasks need to be set which require engagement with context. It is impossible to develop "the modes of thinking which provide ways of modelling situations" (BSSSS, 1992, p. 1) in our environment by continually attempting tasks that avoid any significant engagement with the task context on the part of the problem solver.

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Appendix

Example Tasks

The Shaft Problem (TASK 7): The diameter of a cylindrical shaft is gradually reducing through wear. The rate at which the diameter is changing is given by

$D'(t) = -0.144t^2$ *mm per month for* $t > 0$.

After 1 month of continuous use the diameter is 500.12 mm. How many months will the shaft have been in use before the wear is 1 mm?

Road Construction Problem (TASK 10): A new traffic lane (of minimum width 6 metres) is to be added to a section of highway which passes through a cutting. To construct the new lane, engineers need to excavate an existing earth-bank at the side of the roadway which is inclined at 25° to the horizontal. This will make the inclination steeper. Local council regulations will not allow slopes greater than 40° due to the potential for erosion. Determine if the new traffic lane can be excavated without expensive resumption of the properties at the top of the bank which is 7 metres above the road surface. [List any simplifying assumptions you have had to make.]

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Different forms of mathematical questions for different purposes: Comparing student responses to similar closed and open-ended questions

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Responses of Year 8 students to open-ended and closed mathematics questions, which addressed similar aspects of the curriculum, were compared. The responses of the students were examined and the elements of the tasks were listed. In two out of three pairs examined, the closed question was easier for students. In the other, the open-ended task was easier. It seems that closed and open-ended items may contribute productively both to classroom and specific assessment tasks, although their contribution may be different.

In the past, one of the main roles of mathematics has been as a filter of students wishing to proceed to tertiary studies. The curriculum reflected this orientation, and it was not expected that all students would study mathematics at senior levels nor that all those who did so would master the content. This orientation has now changed. Many more students are choosing to study mathematics at upper secondary levels and many tertiary courses now require an understanding of advanced aspects of secondary mathematics. There is also an expectation by employers, tertiary institutions and the community that graduates are able to use their mathematics in more diverse situations. Mathematics teachers are now striving to identify approaches which will allow them to assist all students to achieve success in mathematics and to provide them with the mathematical background which is necessary for their future studies and careers. Specifically, teachers are now seeking strategies, which allow more students to learn mathematics at higher levels, and which also stimulate ability to solve problems and apply mathematics in unfamiliar contexts.

Teachers are faced with a tension between, on one hand, posing tasks which provide opportunities for development of problem solving abilities and, on the other hand, addressing the content specified within syllabus documents and required for competitive examinations. One approach used to address these dual purposes is the use of content-specific open-ended questions (e.g. Sullivan, Clarke, & Wallbridge, 1991; Sullivan & Clarke, 1991; Sullivan, Bourke & Scott, 1995).

We use the term closed to refer to questions for which there is only one correct response. This is consistent with the approach of Pehkonen (1997). Note that some authors restrict the use of the term closed to multi-choice type items. In our terms, an example of a closed question is:

Find the mean of 8, 10, 12, 12, and 18.

The corresponding open-ended question would be:

The mean of a set of 5 scores is 12. What might be the scores?

The open-ended question draws on the same content, but allows the possibility of the students investigating the situation for themselves and so coming to a better appreciation of the concept of mean as a result of their own thinking. It also produces quite different classroom interactions in that the students can report on their own insights and the variety of solutions they find. With the closed questions, their responses are either right or wrong. There are also concerns with regard to using closed questions for assessment purposes.

responses are either right or wrong. There are also concerns with regard to using closed questions for assessment purposes.

For example, Ellerton and Clements (1997), when using a short-answer (multiple-choice) pen and paper test found a 28% gap between student performance and student understanding. Students who lacked adequate understanding of the key concepts being tested gave correct answers, or students with full or partial understanding gave incorrect answers. These validity concerns support the investigation of the use of open-ended questions for assessment.

Curriculum developers have recently endorsed the use of open-ended questions. For example, the NSW Board of Studies (1996) provided teachers with many examples of sample open-ended questions, and such questions are now used in published curriculum materials (e.g., Sattler & White, 1997). This project seeks to examine the characteristics and effectiveness of different types of questions. Specifically it attempts to examine:

Are content-specific open-ended questions:

i) easier or more difficult than comparable closed questions?

ii) more likely to elicit higher level responses if the prompt is phrased within a practical context?

iii) more likely to elicit higher level responses if explicit prompts for multiple responses are used?

iv) able to be scored validly and reliably if used within an assessment context?

This paper focuses on the first of these questions. Content-specific items used in the project involved area and perimeter and decimals, although only data for area and perimeter are reported at this time.

The approach to data collection

Responses were sought from approximately 1200 students from schools in each of the three Australian states. Data were also collected in Indonesia but these are not reported here. In each case more than two schools responded to the tests. Students were selected to maximise comparability by gender mix, socio-economic status, urban location, school size, and experience of the teachers.

Eight instruments requiring written responses were developed. Each differed according to the mix of open-ended/closed and context/context free. The test forms also sought information on the use of appropriate cueing for multiple responses, the impact of the use of contexts in the phrasing of the questions, and the potential of a mix of open-ended and closed questions on assessment tasks scored holistically. Where possible the items were designed to be comparable with items on large-scale tests used elsewhere. The items were piloted in three states to ensure that there were no difficulties with the wording, and that the content was compatible with the syllabus.

The items on each of the tests were scored individually. The closed items were scored as either correct or incorrect. Each of the open-ended items were scored using the following codes:

i one or two correct responses
ii some correct, some incorrect
iii three or more correct responses

The various forms of the tests were administered to parallel streams across schools and states. At each of the schools the tests were given on the same day.

Comparing responses of students to open-ended and closed questions

For this paper three pairs of open-ended and closed items are compared. The same group of students answered each pair of items. The open-ended items used a prompt "give at least three answers" since during trialling we observed that many students were reluctant to give more than one response. The purpose of each pair of questions was first, to gauge the effect of using differing units of measure, second, to ascertain children's' understanding of the interrelationship between perimeter and area, and third to determine children's' understanding of the relationship between embedded rectangles

Different units

The items requiring the use of different units were designed to be comparable to the following item taken from the Department of Education (1991).

> *A rectangular rug has an area of 2 square metres. The rug is 40 centimetres wide. How long would it be?*
>
> *5 m* *8 m* *20 m* *50 m*

The Department of Education (1991) study reported that 52% percent of year 9 students were able to answer this correctly. The results for the present study are shown in Table 1.

Table 1: Responses to *different unit* items

Item	Type	Response Code	% of correct responses	n
This rectangle has an area of $2m^2$. It is 40cm wide. How long is it?	Closed	Correct	7	226
A rectangle has an area of $3m^2$. What might be the length and width of the rectangle? (give at least three answers)	Open-ended	i	8	226
		ii	6	
		iii	9	

Given the facility of the Department of Education (1991) question, the number of correct responses to the closed item (7%) seemed very low. This may be due to the younger year level used for the sample in this project or to the multi-choice selection of responses given in the original question. The multiple choice format for the Department of Education (1991) supports the hypothesis that a number of students simply had to choose between 5*m* and 50*m*, and thus did not really understand the units involved. This supports the Ellerton and Clements (1997) finding.

A total of 23% of students gave one or more correct responses to the open-ended item. It appears from the results that the open-ended item may be easier than the closed item. An examination of the types of responses for each item gives some insight into why the open-ended question seemed easier. The distribution of responses were as follows:

For the Closed item:

7 students converted to 0.4m then calculated correctly
4 students converted to 20 000cm^2
5 students gave the correct answer although the working out seemed incorrect.

For the Open-ended item:

52 students gave one or more correct responses
48 of these gave 3m x 1m as one of their answers
21 gave 1.5m x 2m as one of their responses
most other correct responses used decimals in some way
another 120 responses were correct numerically but with incorrect units

The percentage of students in the last category of the open-ended question (53%) is comparable with the percentage of correct responses to the Department of Education (1991) question, attesting that the need to use compatible units of measure was the main source of difficulty that students experienced. The results indicate that the students experienced little difficulty with the concept of multiple responses. Even though there were still comparatively few responses to the open-ended item, there were more students who were able to explore the context with the open-ended prompt, with many of these going beyond the trivial 3 x 1 response.

Based on these responses, and on an analysis of the questions, each item was broken into the components needed for success on each. Figure 1 summarises the components. The components common to both items are recorded in the middle column of the table.

For Closed item	For Both	For Open-ended item
Match information to dimensions	Read	Select 2 numbers with product of 3
Convert m^2 to cm^2 OR cm to m	Comprehend	Recognise multiple possibilities
Do division	Encode	
	Record	
	Area is L x W	

Figure 1: Components for each item.

For these two problems, it is possible that the closed item is more difficult because of the need both to recognise the need to convert from one unit to the other and then to do just that. Problems with the units may impede students from exhibiting their understanding of the area concept. The open-ended item does not necessarily require unit conversion, and it seems that those students who responded correctly did so without converting. Hence, the open-ended item gave more information about student's understanding of area but gave little information about their facility with unit conversion. The use of an appropriate prompt may be needed to elicit this information.

In the classroom, the open-ended item may be useful for initial exploration of the area concept with either non whole number sides or mixed units, and the relationship between differing units of measure.

Perimeter and Area

The items related to Perimeter and Area were designed to build on the following item taken from the Department of Education (1991).

What is the area of a rectangle 8 metres long and 3 metres wide?

In that report, 67% of the year 7 students responded correctly.

These items were also related to the following item from Sullivan & Clarke (1991).

A rectangle has a perimeter of 30m. What might be the area?

In that study, 31% of year 6 students and 80% of year 10 students gave one or more correct responses. The pair of items used for this study is shown in Table 2:

Table 2: Responses to *perimeter and area* items

Item	Type	Response Code	% of correct responses	n
A rectangle is 8m long and 5m wide. What is the perimeter and area of the rectangle?	Closed	Perimeter	87	137
		Area	67	
If a rectangle has a perimeter of 30m, what might be the area of the rectangle? (Give at least 3 answers)	Open-ended	i	9	137
		ii	4	
		iii	43	

In this case the closed item elicited more correct responses. Most students also exhibited a higher facility in finding the perimeter of a rectangle as compared with finding the area of the rectangle. This was consistent across all states and schools. The area response was similar to the item. For the open-ended question, 56% of the sample could offer one or more correct answer. The following shows a detailed breakdown of the responses given.

For the closed item:

For perimeter
    49 students used 5 + 5 + 8 + 8
    38 students used (5 x 2) + (8 x 2)
    1 student used (5 + 8) x 2
    32 students were scored correct but did not show working
For area
    45 students used 5 x 8 with correct units
    22 students used 5 x 8 by did not specify units (and were scored as correct)
    25 students were scored as correct but did not show working

For the open-ended item:

2 students (only) used 7.5 x 7.5
all other correct responses used one or more of the whole number possibilities

Again students avoided the use of non-whole number units of measure. The components of the respective items are shown in Figure 2.

For Closed item	For Both	For Open-ended item
	Read	P and A are linked
	Comprehend	Reverse P calculation
	Know P	Do 1 calculation
	Know A	Recognise alternative possibilities
	Calculate	Further calculations
	Record	

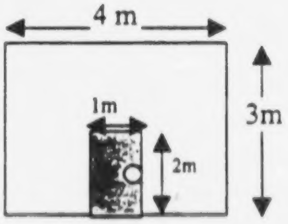
Figure 2: Components of the different items.

The open-ended item requires more integration of ideas (e.g. perimeter and area are linked) which could account for its lower facility. Perhaps in this case, the open-ended item would be better used after students have had experience with a variety of closed items. However, note that the skills required for the open-ended item are desirable and that the open-ended item provides the opportunity to build effective links between units, area and perimeter. Asking for (say) 5 answers could have better assessed the existence of links with units so that fractional lengths would be required.

Embedded rectangles

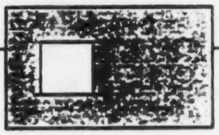
The items related to embedded rectangles were designed to build on the following item taken from the Department of Education (1991).

I want to paint the wall but not the door. What area of wall do I need to paint?



In that report, 49% of the year 7 students responded correctly. The results for this study are shown in Table 3.

Table 3: Responses to *embedded rectangles* items

Item	Type	Response Code	% of responses	n
What is the area of the shaded part of the rectangle? (same diagram as above – but with rectangle and not the door shaded)	Closed	Correct	77	216
The area of the shaded part of this diagram is 8m^2. What might be the dimensions of the large and small rectangles (Give at least 3 answers) 	Open-ended	i ii iii	12 15 24	216

The above results indicate that again students gave more correct responses to the closed item. This was consistent across all states and schools. The distribution of responses to the open-ended and closed questions is shown below.

For the closed item:

136 students used (3 x 4) – (1 x 2) using correct units
17 students gave a correct answer with no working out shown
11 students gave the correct number but used no units
2 gave the correct answer, but seemed to have incorrect working

For the open-ended item:

260 correct responses used whole numbers dimensions
19 correct responses used decimals
9 responses were partially correct
45 students made no attempt

Twenty percent of students gave no response to the open-ended item. The preference for whole number units is again evident. The components for the respective items are presented in Figure 3.

For Closed item	For Both	For Open-ended item
	Calculate area	See need for trial and error
	Recognise shape as one rectangle taken away from another	Recognise multiple possibilities

Figure 3: Components of the *embedded rectangle* items.

As with the preceding pair of questions, the open-ended item appears to have some additional complexity. The 20% of non-attempts for the open-ended version would seem to suggest that the task of subtraction of areas is an obstacle. However, the high facility for the closed item suggests that this does not seem to have been a major problem. For the open-ended item, it may be that the mental effort required for using trial and error to find correct combinations could be a barrier to some students.

Conclusion

The results from this research give some insights the use of open-ended and closed questions. In two out of the three pairs students found the open-ended questions more difficult. An analysis of the types of responses, and summary of the components of the tasks, suggested that the second and third open-ended questions required thinking above and beyond that required for the corresponding closed question. The nature of this thinking needs to be further explored, as it may be that this type of thinking is what we value most in mathematics. Those open-ended items also required students to understand the links between two concepts and to be able to use such links to conjecture and generalise. It may also be that such questions are more appropriate to use once students have some understanding of the concepts involved, including experience with similar closed questions. The open-ended questions could then serve the role of stimulating students' thinking to higher levels. It also seems that an analysis of the components of similar tasks can indicate their comparative difficulty.

Two ancillary results were also interesting. Students seem to be more able to answer perimeter questions as compared with area questions. At Year 8 it seems that most students would have been exposed to both concepts. That one fifth of students could solve the perimeter and not area item is a puzzle.

Second, the use of mixed metric units in the problem seems to make it more difficult to answer. The facility for that item was very low. Perhaps students are seldom exposed to questions for which they have to combine two different aspects of the curriculum.

As opposed to the closed questions, which often show what students do not know, responses which contained correct and incorrect answers are useful in assessing both what students can do and what they cannot do. However, only correct responses to open-ended questions give insights into what students know but may not show what they do not know. Appropriate prompts may be needed to elicit this information.

These results overall will contribute to our understanding of the ways students approach such questions, and how open-ended questions can contribute to a broader assessment routine.

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Student Choice of Computation Methods

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This study was designed to refine an instrument to explore the relationships between students' mental, written and calculator strategies of computation. A theoretical framework for studying the relationships between these three main methods of computation was postulated. This paper describes this model of the computational processes and some preliminary work carried out with middle to upper primary students in Western Australia. In addition, the questions of factors such as access to calculators, emphasis on written algorithms and mental computation influencing choice of computation method are also discussed.

Introduction

The current focus on numeracy and the 'numeracy benchmarks' indicates that there is still a strong focus on number and in particular mental computation and written algorithms, not only in Australia but also elsewhere. The calculator, while it has been widely available in educational settings for well over two decades, often comes in a poor third to the computation mainstays of mental and written calculation. Surveys of teachers' attitudes (Warren & Ling, 1995; Sparrow & Swan, 1997; Jones & Tanner, 1997) on the use of calculators in the nineties indicated that resistance to calculator use in classrooms is quite pronounced. When calculators are used in primary mathematics it is often for checking work or in other low-level contrived situations. There appears to be a reluctance on the part of teachers to allow easy access to the use of calculators for fear that students might become too reliant on them, or that they might make inappropriate choices as to when and how to use them. The Calculator Aware Number (CAN) project carried out in the UK, however, found the opposite to be true (Shuard, Walsh, Goodwin & Worcester, 1991). Children often prided themselves on being able to perform a calculation mentally rather than resort to the use of pen and paper or calculator. Ruthven (1995) made this observation about children who participated in the CAN project.

> The project children were better able to tackle real-world problems and computational tasks; in particular, . . .while they did not make more use of calculators, they made more appropriate choices of calculating device and were better able to interpret their answers.(p. 239)

This raises questions as to what constitutes an appropriate choice, but we do not intend to debate this issue here, other than to state that appropriate choices will vary from child to child, depending upon aspects such as background and experience. For example, we would suggest that computational choice is skewed in favour of written methods due to the strong emphasis on written computation in primary school and the amount of time spent on teaching formal, written algorithms (Porter, 1989).

At first it may appear as though the choice of computational approach is relatively simple and clear-cut but, in reality, many complex factors impinge on choice. For example, in studying the computational choice of children in the marketplace and at school Carraher, Carraher and Schliemann (1985) found that the choice of computational method was influenced by setting. To suggest that computational choice lies between three alternatives is too simplistic. Ruthven (1998) who studied the use of mental, written and calculator strategies by upper primary students noted that ". . . a refinement of the common-sense trichotomy between mental, written and calculator methods was necessary to take better account of different forms and functions of writing within computation (pp. 29-30).

Theoretical Framework

When faced with a mathematical problem, a person must at some point determine whether or not a calculation is required. Given that a calculation is required, the problem solver must then determine whether an exact or only an approximate answer is needed. Often an estimate is sufficient to answer the question. For example, in order to solve the problem, "Do I have enough money to pay for this trolley-load of groceries?", all that is usually required is an estimate. If a problem requires an exact answer then there are several approaches that might be adopted—

- mental computation;
- pencil and paper (written computation);
- a calculator or computer; or
- some combination of these.

There are several influences that impinge on the decision about which calculation method is used. For example, the nature of the numbers may influence how a person plans to perform the calculation. Ruthven (1995) argued that student attitudes and beliefs will also affect decisions about how to tackle a calculation. Those that have concerns about the legitimacy of using a calculator to solve computational problems may well choose to use an alternative method.

The National Council of Teachers of Mathematics (NCTM, 1989) presented the model shown in Figure 1 as a means of describing the process a person goes through when deciding how to tackle a problem which involves a calculation. Trafton (1994) noted some problems with the NCTM model and modified it to emphasise the role of estimation. The NCTM model tends to be linear in nature, suggesting the use of a single method at any one time, whereas in reality a combination of the above three methods may be employed. Also, estimation should be used in conjunction will all three methods.

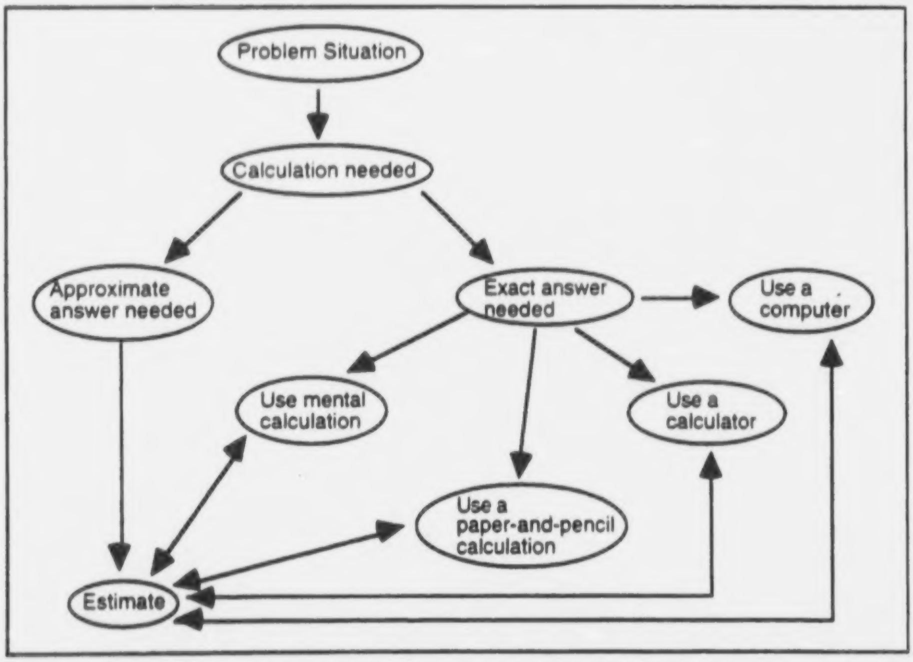

Figure 1. Model to describe computation choice (NCTM 1989, p. 9)

Bobis (1991) linked the use of calculators with number sense suggesting that "students should be taught to check the appropriateness of a computation in three stages — the estimation stage, the calculation stage and the checking stage" (p. 4). Shigematsu, Iwasaki, and Koyama (1994) related mental computation to calculator use with a "metacomputation" framework. They believe that "mental computation will become more than merely a basic skill for paper-and-pencil computation but rather a metacomputation for computation by the calculator" (p. 29). Shumway (1994) expanded on this notion by suggesting that metacomputation might be described as "involving processes and strategies employed to guide computational choices" (p. 194).

Swan and Sparrow (1998) incorporated the work of Bobis (1991) and others in emphasising the ideal of the person involved in a calculation actively monitoring the calculation from any initial estimates right through to checking whether or not the answer is 'sensible'.

> A key aspect of this approach is the involvement of the student throughout all stages of the calculation. While performing the calculation ongoing monitoring should take place. The person carrying out the calculation with the aid of a calculator should not simply push buttons but should monitor each phase of the calculation. (p.166)

In an attempt to better understand the computational process, and in particular the role played by calculators, Bana and Swan (1997) developed a non-linear model designed to take into account some of the complexity of this process. This computation model shown in Figure 2 describes the interplay between the methods of computation. The model also takes into account factors, such as metacognitive strategies, which impinge on calculator use and the computation process generally.

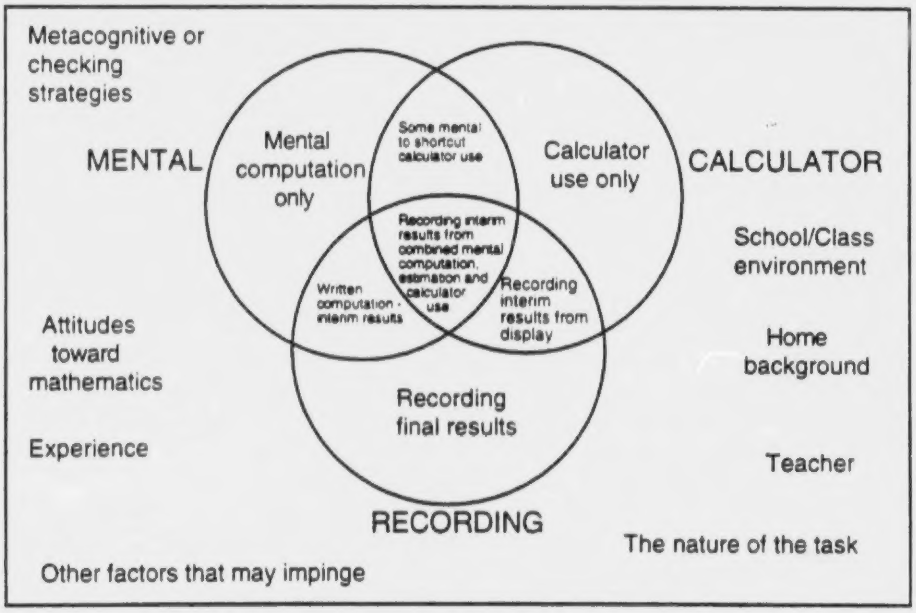
Metacognitive or checking strategies
MENTAL
Mental computation only
Some mental to shortcut calculator use
Calculator use only
CALCULATOR
Recording interim results from combined mental computation, estimation and calculator use
Written computation - interim results
Recording interim results from display
School/Class environment
Attitudes toward mathematics
Home background
Recording final results
Experience
Teacher
RECORDING
The nature of the task
Other factors that may impinge

Figure 2: A model of computational processes (Bana & Swan, 1997)

A brief description of the model in Figure 2 follows. The use of the three primary processes of *mental computation*, *calculator use*, and *recording* seems to provide a sound basis for a model of the computation process. The term *mental computation* refers to all work done in the head when performing a calculation. This includes exact mental computations, estimating and checking solutions, and monitoring strategies applied throughout the calculation. *Calculator use* refers to the entering of data and operations into a calculator, the use of calculator memory facilities, the use of a computer in a computation exercise, and the interpretation of displayed results. The term *recording* denotes any informal jottings during the calculation, as well as the recording of more formal algorithmic steps.

The complex nature of computation is such that, apart from mental computation, most computation activity will involve some combination of two or three of the primary processes identified above. For example, written computation is not seen as a unique process, as it generally involves mental computation as well as recording of interim and final results. As more is learned about the computation process the model will be refined to reflect more closely what actually takes place.

The Study

An exploratory study was undertaken to refine an instrument for the main study which will attempt to test the suitability of the theoretical model in Figure 2. A sample of 12 students from Year 5 to Year 7 from two schools with similar profiles were interviewed individually as they attempted a set of 15 computation items. The interviewer and the student sat together at a table that contained pen, paper and a calculator. At the beginning of the interview, students were told that they could solve the problem using whatever method they liked. Once they had attained an answer to the item, the interviewer asked the students why they chose a particular approach to solving the problem and asked them to explain how they went about solving it. Each item was treated in this way. Field notes were taken as the students attempted to solve the problem, and also later when they explained their strategies and processes.

Some General Observations

The preliminary data from our exploratory study are already yielding some very interesting results. From this limited data the following tentative observations have been made:

- Given a free choice of computational method, most students tended to favour mental computation, though this may have been due to the nature of the items. For example, in the item that involved adding 28 and 37, all students chose to solve this problem mentally.
- Students often chose to use the calculator in questions involving "big numbers" and those that involved decimals, or in the words of some students, "because of the dot".
- Items involving simple fractions proved to be very difficult. For example, The children in the sample struggled with the question two-thirds of forty-five. Some indicated that they had little understanding of fractions, having struggled with them in class.
- Some students changed their approach part way through a calculation. This occurred several times, particularly when they got to a point where they realised they had reached their limitations. The item 23·35+34·75 was one example where this happened. Students often tried the calculation mentally, got stuck, and then reached for a calculator.
- Items involving 'carrying' or regrouping were more likely to be completed with pen and paper.
- Several students used a combination of mental methods and informal jottings.
- When students began to run into difficulties or realised they had made a mistake they would often turn to the calculator.

- Some students made use of the calculator only as a checking device.
- All cases appeared to fit the theoretical model of computation processes illustrated in Figure 2.

Data from Selected Items

The following are examples of observations from some of the 15 items used in the exploratory study.

Item — 23·35 + 34·75: One student started to answer this question mentally, realised it involved 'carrying', then chose to use a calculator to complete the question. When probed a little further the student was unsure whether or not the answer produced with the aid of a calculator was correct. He commented that he wasn't "too sure on my points". Another student used the calculator to correctly answer the question and stated that she "hated decimals". Another student who commented that she was "not that good at decimals" used the calculator, obtained the correct result and commented, "that looks about right", after performing the mental calculation, 23 + 34.

Item — 7·41 – 2·5: This item caused some concern for students. One student said she couldn't do it because there were two digits behind the decimal in the first part and only one in the second part. She then went on to use the calculator and found the correct answer.

Item — 1·5 x 20: None of the students used pen and paper methods for this item. One child who used a calculator and got an answer of 30 was surprised indicating that he had expected to get 21. This tends to indicate that he had thought about a possible result prior to the answer appearing on the display. Students using mental methods instantly recognised that 0·5 was one-half, knew that half of twenty was ten, and added ten to twenty to make thirty.

Item — Half of 5 times 2: One student initially chose to use a calculator, then tried a combination of mental computation and calculator but still could not complete the calculation.

Item — 3000 x 4000: Many students chose to complete this problem with the aid of a calculator, which was somewhat surprising given the relatively simple nature of the problem. All those using a calculator explained it was because the numbers were big. Students who chose to use mental computation methods invariably took off the zeros then added zeros. This method did not always produce a correct answer, and when probed it was clear that students often seemed to apply this rule with little or no understanding.

Item — 14 x 9 ÷ 6: This item elicited a number of interesting responses. For example, one student used a formal written algorithm to calculate 14 x 9 and then completed 156 ÷ 6 on the calculator. The student said that the second part was done on a calculator because it involved division. After completing the problem this way the student then used the calculator to work out 14 x 9 and found it to be 126 rather than 156, and then completed the calculation with the aid of a calculator. When probed as to why she had chosen to repeat the calculation her response was vague and didn't indicate any checking procedures such as preliminary estimates that had caused her to redo the calculation. One student used a combination of all three methods. Firstly she mentally determined that 12 x 9 was 108 and then worked out 2 x 9 and added the two results to arrive at the answer of 126. The rest of the calculation, 126 ÷ 6, was completed using pen and paper. Finally the answer was checked using a calculator. A similar technique was used by another student. After completing 14 x 9 mentally the student attempted to complete 126 ÷ 6 using a formal written algorithm, but experienced difficulties and then reached for the calculator to complete the problem. This item certainly caused students to rethink their method of computation. Some realised they were in difficulty or had made a mistake but had problems explaining how they knew they had made a mistake.

Item — 1 x 2 x 3 x 4 x 5 x 6 x 7 x 0: It appears that students often do not take the time to look at a question before making a choice of computational process. Several students initially reached for the calculator, got part way through the solution and abandoned the calculator in favour of mental computation. One student who started the calculation using paper and pencil methods became totally lost as the numbers became larger and gave up before completing the problem. She then opted to use a calculator and discovered that the answer was zero and said, "that's different".

Items — 99c + $2·39; $6·99 + $2·39 + $1·79: These two in-context items provided some interesting data. Students were given some advertisements for chocolate, biscuits and soft drink that were cut from the local paper and asked to calculate the total cost if buying certain combinations of items. A block of chocolate was advertised at $2·39, biscuits at 99c, a carton of soft drink at $6·99 and one bottle of soft drink at $1·79. The first question, 99c + $2·39, was mainly answered using a mental computation method that involved rounding the 99c to one dollar, performing the addition and then compensating for the one cent. Similar reasoning was used to complete the second question with the $6·99 being rounded to $7·00, the $2·39 to $2·40 and the $1·79 being rounded to $1·80. One student completed the problem in written form. Interestingly, no-one used a calculator.

Item — 300 ÷ 6: This item invoked an interesting response from one Year 6 student who commented, "on a test I would do it on paper because I know I am not cheating myself. Using a calculator is a bit like cheating because you don't know the answer until it comes up on the screen". It should be noted, however, that this student eventually used a calculator to attain the correct result.

Where to from here?

Several lines of inquiry have resulted from this exploratory study. The authors are currently gathering further data to evaluate the model postulated above. Of particular interest are the various mixes and blends of mental, written and calculator methods of computation. The aspect of switching from one form of calculation to another is a further avenue to be investigated. The instruments used during the pilot have been refined as appropriate and data collection is under way in the main study. A number of questions are being addressed, and these are listed below.

What is it about a particular question that triggers a child to reach for a calculator? There is some suggestion that large numbers, decimals and fractions may be triggers to pick up a calculator, but clearly there is a difference between an item such as 3000 x 4000 and another like 2387 x 9473, even though we may classify both as four-digit by four-digit multiplication.

What monitoring strategies, if any, do students apply when performing a computation? It appears that, while students may not apply explicit monitoring strategies, there are certain techniques that some students use to check their calculations.

Do computational 'tricks' such as 'take off the zeros' help children or confuse them? McIntosh, De Nardi and Swan (1994) have already noted that "teaching rules (for example 'removing zeros') is as dangerous and self-defeating for mental computation as it is for written computation" (p. 4). Clearly this was the case with the pilot group. Unfortunately onlookers who observe a child using a calculator to complete a computation such as 3000 x 4000 look on with disdain and make the claim that the child is becoming reliant on the calculator to do simple problems.

Do students who have been exposed to a calculator-rich environment tackle problems in a different way from students in calculator-impoverished environments? Indications from the CAN project (Shuard, 1992) are that students have been influenced considerably by their exposure to a calculator-rich environment. At the very least, such students have a good knowledge of how to use a calculator and the limitations of the calculator. Students in this study did not choose to use a calculator to tackle a problem such as $^{2}/_{3}$ of 45. This may have been the case because they did not realise that you could multiply 45 by 2 and then divide by three to obtain an answer.

Does the availability of a calculator influence the way students might think about tackling a problem? Fielker (1992) and Baggett and Ehrenfeucht (1994) refer to the notion of "calculator algorithms" or plans used to solve problems with the aid of a calculator. Often these plans are expressed as keystroke sequences. Fielker has coined the term "calgorithm" to describe this idea. No students in the present study displayed the type of behaviour discussed by Fielker and by Bagget and Ehrenfeucht, but some children in the study chose to use mental computation except when it came to division, in which case they ignored written computation in favour of the calculator.

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Mathematical misconceptions
-- we have an effective method for reducing their incidence but will the improvement persist?

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This paper discusses a continuation of the study previously conducted by these authors in which a strategy based on Piaget's notion of cognitive conflict was successfully employed to reduce the incidence of mathematical misconceptions in a group of tertiary students. A question which remained unanswered in the authors' previous study was "will the improvement persist or have we seen a short term gain which will disappear in time?" The students were tested again one year later to decide this issue.

Introduction

The variety and importance of mathematical misconceptions exhibited by students, and ways to reduce or eliminate them, has been studied by a number of researchers (Bell, 1982; Farrell, 1992; Margulies, 1993; Perso, 1992). Some of them believe that by considering the mathematical misconceptions which are exhibited by students, teaching techniques can be developed which aim at diagnosing and eliminating those misconceptions. Davis (1984) suggests that the number of different types of mathematical misconceptions is enormous, "and a complete list may not even be practical" (p. 335). Swedosh (1996) discussed the types, frequencies of occurrence, and possible reasons for the mathematical misconceptions commonly exhibited by mathematics students on entering tertiary mathematics subjects at the University of Melbourne (U. of M.) and at LaTrobe University (LaTrobe).

There have been various attempts to eliminate students' misconceptions in a wide range of fields. One approach which has met with some success is the "conflict teaching approach", based on Piaget's notion of cognitive conflict, in which a teacher and a learner discuss the inconsistencies in the learner's thinking so that the learner realises that the conceptions exhibited were inadequate or faulty and needed modification (Tirosh, 1990). Vinner (1990) argues that "there is no doubt that if inconsistencies in the students' thinking are drawn to their attention, it will help some of them to resolve some inconsistencies in a desirable way" (p. 97). The conflict teaching approach has been found to be effective in successfully resolving misconceptions relating to aspects of mathematics and physics (Stavy and Berkovitz, 1980; Strauss, 1972; Swan, 1983).

Swedosh and Clark (1997) reported on an experiment devised to test whether the conflict teaching strategy helped to reduce the frequency of mathematical misconceptions in a sample of mathematics students at the U. of M. The strategy proved to be extremely effective and the improvement greatly exceeded the expectations of the authors.

> "It is clear that by first challenging or undermining the misconception held by the students by showing the ridiculous outcomes which can flow from such 'rules', and then replacing the 'damaged' concept with the correct one, mathematical misconceptions can be, to a great extent, eliminated" (p. 498).

The study by Swedosh and Clark (1997) demonstrated that the conflict teaching approach was found to be effective in significantly reducing, and in some cases eliminating, mathematical misconceptions. The study did, however, raise some other questions, namely, would the strategy be as successful with less able students, would the strategy be as successful if the concepts being tested were embedded in more complex questions, and would the improvement persist or just be a short term phenomenon with students reverting to their previously held conceptions. It is this latter question which is examined in this study. The question of the effectiveness of this method with less able students will be addressed during Semester 1, 1998, when an experiment will be performed at the U. of M. using the conflict teaching approach on an appropriate group of students; the results of this study will be available at the MERGA Conference in July, 1998.

The importance of eliminating mathematical misconceptions, and arriving at a complete working understanding of basic concepts, is well documented and is especially important if students are intending to continue their studies in mathematics. The issues of a sound preparation for further studies and of increasing participation in post-secondary mathematics education have been officially recognised by the Victorian Government (Ministry of Education Victoria, 1984; Victorian Government, 1987) and in A National Statement on Mathematics for Australian Schools (Australian Education Council, 1990). One goal in the National Statement is that "as a result of learning mathematics in school, all students should possess sufficient command of mathematical expressions, representations and technology to continue to learn mathematics independently and collaboratively" (p. 18).

In many branches of mathematics, success at a particular level has been shown to depend heavily on students having previously achieved mastery of a limited number of basic concepts and being able to confidently use certain skills (Swedosh, 1996). In his paper, Swedosh also asserts that the preparedness of students to study tertiary mathematics is largely determined by the level of understanding of the basic mathematical concepts which are expected to be acquired in secondary school. Since subsequent learning of mathematics relies to such an extent on the mastery of pre-requisite knowledge, there has been a high level of interest amongst mathematics educators (teachers, lecturers, and tutors) regarding mathematical misconceptions, their frequencies of occurrence, and whether a method exists which effectively reduces these frequencies.

The issue of preparedness of students for tertiary mathematics takes on even greater significance when one considers the statements by Blyth and Calegari (1985) that "contrary to a widely held belief, 90% of all HSC students do apply for tertiary entrance. It is reasonable to infer from this that most students see HSC as a preparation for tertiary studies" (p. 312) and also, "statistics collected by the Mathematical Association of Victoria (MAV) show that 75% of all tertiary courses require a pass in HSC mathematics" (p. 312). These statements were made in 1985, and there is little to suggest that the situation has changed dramatically since then.

Given the negative impact that mathematical misconceptions can have on the mathematical future of students, and that the conflict teaching approach had been shown to be successful in reducing their frequencies, the next step was to investigate whether the improvement which had resulted would persist or whether the effect was temporary (that is, students would revert in due course to displaying the same misconceptions as they had previously). This study reports on our attempt to clarify this issue.

Methodology

The methodology for this study could be divided into two parts, the first part being the methodology from Swedosh and Clark (1997) in which the students were given a pre-test and a post-test, and the second part being the re-testing of a subset of the original group of students 12 months later. In each case, students who studied Applied Mathematics in their first year at the U. of M. in 1996 were involved. Each test was made up of questions similar to those used in earlier tests at the U. of M. and at LaTrobe (Worley, 1993) in which students had previously exhibited a high frequency of misconceptions (Swedosh, 1996) and some questions similar to those appearing in a list of questions in 'Algebraic Atrocities' (Margulies, 1993, p. 41). Each question was designed in such a way that if the student had a particular misconception, this would become apparent when considering the response of that student to the question.

The first test was called the pre-test as it was held prior to using a teaching strategy designed to reduce the frequency of misconceptions. This test consisted of 53 short answer questions. Students were given sufficient time to complete the test. Each response to each question on each student's paper was carefully examined to gain information on any misconceptions which had been exhibited. A tally was kept of the number of students who had attempted each question, how many had answered the question correctly, how many had exhibited a misconception, and how many had given another wrong answer. The questions in which students exhibited the highest frequency of misconceptions became the 'focus' questions of this study; there were 17 such questions. It was felt that these 17 questions could be completed by students in 10 minutes, and that this interval of time would not impinge too greatly on the lecturer's class time.

Two weeks after the pre-test, about twenty minutes of a lecture was dedicated to using the 'conflict teaching approach' in an attempt to eliminate or reduce the frequencies of these misconceptions and then to replace them with the correct concept. For example, 6.7% of students had exhibited the misconception that given $\frac{1}{x}-\frac{1}{b}=\frac{1}{a}$, then $x-b=a$ and therefore $x=a+b$. The approach used was to display to students an equation to which the result was self-evident, such as $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$. Using the misconception would lead to the absurd result that $2-3=6$. The correct procedure, in which a common denominator was used, was taught after showing the absurdity which stemmed from the misconception. The rationale in this method is that having seen that the previously held conception simply does not work, the student will happily discard it and embrace the correct concept.

Three weeks after the teaching session, the same class of students sat the post-test which comprised the 17 focus questions. The delay between the teaching and the post-test was important so that students were answering questions not based on remembering the actual examples they had been shown, but on what they now understood to be the correct concepts. Students were not informed in advance that there was to be a second test and so the results of this test could not be affected by any attempt to prepare for it.

For each question, a tally was again kept of the number of students attempting the question, how many had answered the question correctly, how many had exhibited a misconception, and how many had given another wrong answer, and a comparison was made between the frequency of the misconceptions on each question before and after the teaching session. A great improvement was found, with many less misconceptions being exhibited on the post-test. One question which begged for further investigation was whether this improvement would last, or whether, with time, students would revert to their previous, and often long held, misconceptions.

To address this issue, it was decided that about one year after the earlier testing, the students who had been part of the original study (Swedosh and Clark, 1997) would, if they could be contacted and were agreeable, be tested again using the same 17 questions. The students were contacted and asked to come in and sit a test to assist with a study being undertaken. Each student was sent a letter inviting them to participate in the study, and this was followed up by email, telephone, or further letters as appropriate. The students were not told what the test covered or what it hoped to achieve, only that we would appreciate them giving up ten minutes or so of their time to assist with our study. Some of these students were studying a variety of second year mathematics subjects and some were no longer studying any mathematics, but it was hoped that a representative number of those invited would accept due to the considerable good will established during the previous year. Of the 60 students who had been part of the original study, 28 accepted our invitation to be tested again one year later in October 1997. Readers of this paper are referred to the 1997 paper of Swedosh and Clark for a detailed discussion of the background of students who were part of the original study. Details provided include a description of the mathematics subjects studied at secondary school by those students, the assessment methods and grades awarded for those subjects, and an explanation of the Tertiary Entrance Ranking or TER. It can be seen from the statistics in Table 1 that the groups of 60 and 28 students have very similar average scores in mathematics and in their TERs and can be considered to be comprised of very comparable students.

Table 1

STUDENTS IN SAMPLE	AVERAGE SCORES			
	TER	SPECIALIST MATHS		
		CAT 1	CAT 2	CAT 3
60 students who sat pre-test and pre-test:	93.56	9.38	8.52	8.50
28 students who sat the test one year later:	93.34	9.39	8.71	8.43

The test

The 17 questions in the post-test and in the test sat one year later are shown below, and the most common misconception(s) are shown to the right of each question :

Simplify expressions 1-5 as fully as possible. Note that some expressions may not be able to be simplified. If this is the case, simply rewrite the expression in the space provided for the answer.

1.	$\frac{100!}{98!}$	$2!$
2.	$\left(\frac{1}{2}\right)^{-3}$	$\frac{1}{8}$
3.	$3^x \times 3^x$	9^{2x}; 3^{x^2}
4.	$2^x + 2^x$	4^x
5.	$a^b \times a^c$	a^{bc}

Solve equations 6-12 for x :

6.	$x^2 = 81$	$x = 9$
7.	$x^2 - 4x = 0$	$x = 4$
8.	$(x-1)(x^2 - 3x) = 0$	$x = 1$; $x = 1, 3$
9.	$x^2 = x$	$x = 1$
10.	$\frac{x^2 - 1}{x - 1} = 0$	$x = \pm 1$; $x = 1$
11.	$\frac{1}{x} - \frac{1}{b} = \frac{1}{a}$	$x = a + b$
12.	$\log_e x = \log_e t - t + c$, c constant	$x = t - e^t + e^c$

13.	Solve for x : $2x+4<5x+10$	$x<-2;\ x>2$
14.	Factorise $(2x+y)^2-x^2$	$3x^2+4xy+y^2$
15.	Given that $\sin\frac{\pi}{6}=\frac{1}{2}$, evaluate $\sin\frac{7\pi}{6}$	$\frac{1}{2}$
16.	Given that $\sin A=\frac{3}{5}$ and that $\frac{\pi}{2}<A<\pi$, evaluate $\cos A$.	$\frac{4}{5}$

For question 17, indicate whether the statement is True or False. (circle one)

17.	$k\left(50-\frac{x}{5}\right)(80-2x)=k(250-x)(40-x)$ T / F	True

The teaching strategy

Teaching designed to reduce or eliminate the misconceptions exhibited in the pre-test was employed two weeks after the pre-test (Swedosh and Clark, 1997). The concepts arising in questions which were to be on the post-test and on the test sat one year later were specifically targeted. The strategy involved demonstrating that the misconception would lead to a ridiculous conclusion and used mainly numerical examples. Having shown the students that a conception was faulty and in need of replacement, the correct concept was illustrated using slightly different examples to the questions on the post-test (different numbers, etc.) so that students could not simply remember what they had been shown, but had to use the concept correctly to answer each question. Some examples of how the strategy was used and how the correct concept was illustrated are shown below.

$$\frac{4!}{2!}=\frac{24}{2}=12,\ (4-2)!=2!=2;\qquad \frac{4!}{2!}\neq(4-2)!$$

$$3^2\times3^2=9\times9=81;\quad 9^4=9\times9\times9\times9=81^2;\quad 3^2\times3^2\neq9^4$$

$$2^3+2^3=8+8=16;\quad 4^3=4\times4\times4=64;\quad 2^3+2^3\neq4^3$$

$$x^2=9,\ x^2-9=0,\ (x-3)(x+3)=0,\ x=\pm3$$

$$\frac{x^2-4}{x-2}=\frac{(x-2)(x+2)}{x-2}=x+2\ \text{ iff }\ x\neq2;\ \frac{0}{0}\ \text{does not exist.}$$

$$\frac{1}{2}-\frac{1}{3}=\frac{1}{6};\quad 2-3\neq6$$

$$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2};\quad \sin\frac{4\pi}{3}=\sin\left(\pi+\frac{\pi}{3}\right)=-\sin\frac{\pi}{3}=-\frac{\sqrt{3}}{2}\ \text{ as }\ \frac{4\pi}{3}\ \text{is in the third quadrant.}$$

Results

The two tables below summarise the answers to the 17 test questions. Tables 2 and 3 show the frequencies and percentages respectively of each category of answer given in the pre-test, the post-test, and after one year. 'Q' is the question number.

Table 2

Q	Correct			Misconception			Other wrong			No attempt		
	Pre	Post	1 year	Pre	Post	1 year	Pre	Post	1 year	Pre	Post	1 year
	n=60		n=28	n=60		n=28	n=60		n=28	n=60		n=28
1	52	58	27	5	0	0	2	2	1	1	0	0
2	56	60	27	4	0	1	0	0	0	0	0	0
3	54	56	27	6	2	0	0	1	1	0	1	0
4	39	47	22	8	2	2	12	9	4	1	2	0
5	52	57	24	7	2	3	1	0	1	0	1	0
6	49	60	26	11	0	2	0	0	0	0	0	0
7	53	59	26	5	0	1	2	1	1	0	0	0
8	53	57	27	6	1	1	1	2	0	0	0	0
9	28	57	24	30	2	4	2	1	0	0	0	0
10	46	44	24	12	9	1	2	6	3	0	1	0
11	56	57	27	4	2	0	0	0	1	0	1	0
12	36	54	22	15	2	5	4	2	1	5	2	0
13	45	50	23	7	4	3	4	6	1	4	0	1
14	41	55	21	7	0	1	10	2	5	2	3	1
15	51	51	26	7	6	0	2	2	2	0	1	0
16	27	50	20	23	5	5	4	4	3	6	1	0
17	44	54	24	10	4	4	0	0	0	6	2	0

Table 3

Of those who attempted the question

Q	% Correct			% Misconception			% Other error		
	Pre	Post	1 year	Pre	Post	1 year	Pre	Post	1 year
1	88.1	96.7	96.4	8.5	0.0	0.0	3.4	3.3	3.6
2	93.3	100.0	96.4	6.7	0.0	3.6	0.0	0.0	0.0
3	90.0	94.9	96.4	10.0	3.4	0.0	0.0	1.7	3.6
4	66.1	81.0	78.6	13.6	3.4	7.1	20.3	15.5	14.3
5	86.7	96.6	85.7	11.7	3.4	10.7	1.7	0.0	3.6
6	81.7	100.0	92.9	18.3	0.0	7.1	0.0	0.0	0.0
7	88.3	98.3	92.9	8.3	0.0	3.6	3.3	1.7	3.6
8	88.3	95.0	96.4	10.0	1.7	3.6	1.7	3.3	0.0
9	46.7	95.0	85.7	50.0	3.3	14.3	3.3	1.7	0.0
10	76.7	74.6	85.7	20.0	15.3	3.6	3.3	10.2	10.7
11	93.3	96.6	96.4	6.7	3.4	0.0	0.0	0.0	3.6
12	65.5	93.1	78.6	27.3	3.4	17.9	7.3	3.4	3.6
13	80.4	83.3	85.2	12.5	6.7	11.1	7.1	10.0	3.7
14	70.7	96.5	77.8	12.1	0.0	3.7	17.2	3.5	18.5
15	85.0	86.4	92.9	11.7	10.2	0.0	3.3	3.4	7.1
16	50.0	84.7	71.4	42.6	8.5	17.9	7.4	6.8	10.7
17	81.5	93.1	85.7	18.5	6.9	14.3	0.0	0.0	0.0
Total	78.6	92.1	88.0	16.8	4.1	7.0	4.6	3.8	5.1

The results indicate that overall the proportion of misconceptions dropped substantially from the pre-test (16.8%) to the post-test (4.1%), and that one year later there was some reversion (7.0%) as expected. Nevertheless, a marked improvement was seen from the first test to the final test both in terms of the decrease in the proportion of misconceptions and the proportion of correct answers given (78.6% to 88.0%).

When considering the proportion of misconceptions on a question by question basis, on every question there was a substantial improvement from the pre-test to the post-test except for questions 10 and 15 in which the improvement was slight; possible reasons are given in Swedosh and Clark (1997). One year later, most of the questions show the same pattern; an improvement from the pre-test to the post-test, followed by a slight 'drift' one year later, though not back to the original levels. The exceptions were questions 1, 3, 10, 11 and 15. Question 10 showed no drift, and the results for the other four questions actually showed further improvement between the post-test and the test held one year later. The reasons for these unexpected improvements have been considered by the authors and are most likely to be due to either chance (28 of a total of 60 students sat the final test), a further year of exposure to mathematics for many of the sample in which they saw the correct concept during lectures and tutorials, or perhaps for some topics or questions it takes longer than a few weeks for the understanding to 'sink in'.

Limitations of the Study

As stated in Swedosh and Clark (1997), it is important to keep the findings of this study in context, as it is not clear that it is valid to generalise them beyond the scope of the cohort examined. The strategy used, based on cognitive conflict, has been found to be very useful with very bright students who were strong mathematically, but any assertion made about other categories of students might not be appropriate. The improvement for students whose backgrounds were not as good may not be as dramatic or may not occur. The reason for this conjecture is two-fold: the students involved in this study were capable of recognising the inconsistencies in their previous thinking and then learning the correct concepts quickly; and these students demonstrated during discussions with the authors that they were embarrassed by making the errors that they did on what they considered to be material of such an elementary nature, and therefore had a strong desire to remedy the situation. Both of these qualities are likely to be more pronounced with better students. As previously stated, the effectiveness of this method with less able students will be investigated by using this strategy on an appropriate group of students during Semester 1, 1998 at the U. of M.

There is also a possibility if the concept being considered was embedded in some longer and less transparent question, misconceptions may be more likely to recur.

Conclusions

The teaching strategy used, based on the notion of cognitive conflict, was extremely effective and the improvement from the pre-test to the post-test and to the test held one year later certainly exceeded the expectations of the authors. It was expected that there would be some degree of drift when students were tested one year later, but the bulk of the improvement persisted. Swedosh and Clark (1997, p. 499) stated that

> "It is clear that by first challenging or undermining the misconception held by the students by showing the ridiculous outcomes which can flow from such 'rules', and then replacing the 'damaged' concept with the correct one, mathematical misconceptions can be, to a great extent, eliminated."

It is also clear that the improvement seen due to this strategy is much more than a short term quick fix with substantial benefits still evident one year later.

This strategy may not be as effective with students who are less mathematically able or perhaps less able overall; this will be the subject of further research in Semester 1, 1998, and this research will be available at the MERGA Conference in July, 1998. Also, if the questions used were more complex and the concepts were therefore tested more subtly, the level of improvement may be quite different.

This study set out to ascertain whether the strategy which was used to eliminate mathematical misconceptions and which was judged to be a remarkable success in the short term (Swedosh and Clark, 1997), would continue to have a positive effect in the longer term in that some or most of the benefit would persist. The results show that while a small proportion of the improvement diminished, a large improvement was still evident one year later and most of the benefit to students had been retained.

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Dynamic scaffolding and reflective discourse: successful teaching styles observed within a project to teach mathematical thinking skills

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The mathematical thinking skills project (Tanner & Jones, 1995) reported that classes which followed a course emphasising metacognitive skills were not only more successful than controls in assessments of those skills, but also in assessments of mathematical development. Ethnographic data revealed significant variations in the teaching style from teacher to teacher and was used to classify the teachers into four groups. This paper discusses the teaching styles of the two most successful groups: the ìdynamic scaffoldersî and the ìreflective scaffoldersî.

Thinking mathematically

Thinking as sense making is deeply embedded in the constructivist viewpoint in which the learner is considered as an active purveyor of meaning (McGuinness, 1993) and within this tradition, a clear distinction may be drawn between mathematical thinking and the knowledge base, strategies and techniques described as mathematics. Learning to think mathematically is more than just learning to use mathematical techniques, although developing a facility with the tools of the trade is clearly an element. Mathematical thinkers have a way of seeing, representing and analysing their world, and a tendency to engage in the practices of mathematical communities.

> Learning to think mathematically means (a) developing a mathematical point of view - valuing the processes of mathematisation and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure - mathematical sense making
> (Schoenfeld, 1994, p60).

Wheeler (1982) claims that it is "more useful to know how to mathematize than to know a lot of mathematics" and suggests that pupils should be taught to function as mathematicians. Whilst mathematical sense making may have its roots in constructivism, developing a mathematical point of view may be more akin to enculturation into a community. Two clusters of positions are identifiable which claim to explain learning in social contexts. According to the constructivist viewpoint ìan individual makes sense of her surroundings, and tests hypotheses and sense making, by means of her actions, and through responses from others (and the environment)î, whereas according to the sociocultural viewpoint ìpupils learn to operate mathematically in contextsî and ìa person's identity in a new context is constructed within that contextî (Dawson, 1994, p24-25).

This can be summarized as being the difference between the individual constructing his/her world by making sense of it, and being constructed by his/her world by participating in it. However the two need not be mutually exclusive and a middle position between individualistic and collectivist perspectives is taken here, in which the teacher and pupils are considered to interactively constitute the culture of the classroom through negotiation and communication (Bauersfeld, 1994). But within the constraints of that negotiated culture, individuals construct. From such a view, pupils should learn by participating in a ìculture of mathematisingî which is characterised by subjective reconstruction of knowledge through negotiation of meaning in social interaction (Bauersfeld, 1988).

Cobb et al (1997, p269) claim that a ìmathematical dispositionî may be developed in an indirect manner through participation in ìreflective classroom discourseî. In reflective discourse, teachers should manage the interplay of social norms and patterns of interaction to create opportunities for pupils to reason for themselves and ìengage in reflective thinking or reflective abstractionî (Wood, 1996, p102-103).

One of the issues which arises is the extent to which the teacher acts as a genuinely, neutral moderator of discussions amongst co-participants or as a director and guide of pupils' learning. There is an obvious power imbalance between teachers and pupils in classrooms and teachers' comments carry great weight. What is significant is the manner in which the power is expressed in action (Cobb et al, 1992, p486).

It is possible to distinguish between two very different forms of interaction which might be described as scaffolding to support pupils' learning: *funnelling* and *focusing* (Bauersfeld, 1988; Wood, 1994). In *funnelling* it is the teacher who is involved in using thinking strategies and carrying out the demanding tasks to lead the discourse to a predetermined solution. The social processes of the classroom hide the mathematical structure, which the pupil may only construct by choosing to reflect on regularities in the actions performed. ìContext- and problem-specific routines and skillsî are likely to result (Bauersfeld, 1988, p37). Mathematical logic and meaning are replaced by the social logic and meaning of the interaction. In *focusing* the teacher's questions draw attention to critical features of the problem which might not yet be understood. The pupil is then left to resolve perturbations which have thus been created (Wood, 1994, p160).

The discourse mode of teaching is claimed to lead to higher levels of understanding and thoughtfulness in mathematics (Prawat, 1991). It is in ìreflective discourseî, that teachers are able to ìproactively support students' mathematical developmentî by guiding and if necessary initiating shifts in the discourse so that ìwhat was previously done in action can become an explicit topic of conversationî and thus ìparticipation in this type of discourse constitutes conditions for the possibility of mathematical learningî (Cobb et al, 1997, p264 - 269). The social character of the discourse may be arranged to lend social status to ìthe disposition to meaning construction activitiesî which is a ìhabit of thoughtî that can be learned (Resnick, 1988, p40).

From a Piagetian viewpoint, adolescence marks the onset of formal thought - the ability to argue from a hypothesis and to view reality as a reflection of theoretical possibilities. Formal thought has been described as a systematic way of thinking; a generalized orientation towards problem-solving with an improvement in the student's ability to organize and structure the elements of a problem (Sutherland 1992). However, these key aspects of problem-solving are metacognitive rather than conceptual in nature. It can be argued, therefore, that formal thought is underpinned by the development of metacognitive skills.

Recent research suggests that cognitive development can be accelerated (eg: Shayer and Adey, 1993; Cardelle-Elawar, 1995). A key feature of these studies has been their deliberate enhancement of metacognitive skills. Indeed, metacognition has been identified by McGuinness (1993) as a primary tool for conceptual development.

Metacognition

Metacognition is a ìfuzzyî and elusive term which is used as an umbrella to cover a range of ill defined interacting categories which share certain family resemblances

(Brown, 1987, p106). It refers loosely to the knowledge and control which individuals have of their own cognitive systems (Flavell, 1976; Brown, 1987). This dual nature includes both (a) the awareness that individuals have of their own knowledge, their strengths and weaknesses, their beliefs about themselves as learners and the nature of mathematics; and (b) their ability to regulate their own actions in the application of that knowledge (Flavell, 1976; Brown, 1987). The former aspect is passive in character and is characterised here as metacognitive knowledge or ìknowing what you knowî whereas the second refers to the ìactive monitoring and consequent regulation and orchestrationî of cognition (Flavell, 1976, p232) and is characterised here as metacognitive skill. The use of the term ìmetacognitive skillsî to distinguish metacognitive activities and processes from metacognitive knowledge is deliberate and consistent with an emphasis on processes that might be improved by training.

The Practical Applications of Mathematics Project

This project (Tanner & Jones, 1994) identified the metacognitive skills of planning, monitoring and evaluating as necessary for successful practical problem solving and suggested classroom practices which would facilitate their development. These included the use of social structures to frame pupils' behaviour and constrain them to act as experts rather than novices, eg by slowing down impulsive behaviour and encouraging the examination of several problem formulations; the development of a discourse in which differences in perspective were welcomed; the use of focusing questions in scientific argument; and the encouragement of reflective discourse through peer and self assessment. One key practice which was developed was referred to as ìStart-stop-goî in which pupils were asked to read the problem and think in silence for a few minutes and then discuss possible plans in small groups before a teacher led brainstorming session which focused attention on key features. At intervals work was stopped for group progress reports to encourage monitoring.

The Mathematical Thinking Skills Project

These classroom practices were taken as the basis of a project to develop and evaluate a mathematical thinking skills course. The thinking skills targeted were metacognitive rather than cognitive. That is the course focused on the processes rather than the content of mathematics in the context of practical problem solving and modelling. It was hypothesised that "near transfer" would be found, meaning that pupils would demonstrate improved performance in modelling situations which were similar in character to those used in the course, but did not repeat the content of the lessons. It was further hypothesised that the development of metacognitive skills would lead to improved learning in mathematics through "far transfer" into the cognitive domain and pupils' performance was assessed in the content areas of mathematics which had not been targeted by the course.

Methodology

An action research network of six secondary schools was established to develop and trial teaching strategies and materials, supported by members of the project team. Two matched pairs of classes were identified in each school to act as control and intervention groups. One pair was in year seven (11-12 years old) and one pair in year eight (12-13 years old). Matched classes were either of mixed ability or parallel sets in every case. Regular participant observation by the university researchers was necessary to record the nature of the interventions made. These observations revealed that the extent to which teachers were able to adopt the suggested approaches was variable.

Three teachers were dropped from the final analysis for failing to follow the course or approaches to any appreciable extent (see Tanner, 1997 for details).

Written test papers were designed to assess pupils' cognitive and metacognitive development based on a neo-Piagetian structure (see Tanner 1997 for details). Items emphasised comprehension rather than recall. The metacognitive skills of question posing, planning, evaluating and reflecting were assessed through a section in the written paper entitled "Planning and doing an experiment". Metacognitive self knowledge was also assessed by asking students to predict the number of questions they would get correct *before* and *after* each section (referred to here as *forecasting* and *postcasting*). The overall test was reliable with Cronbach's alpha of 0.86.

The pilot course and intervention teaching lasted for approximately five months. Regular network meetings were held at which experiences were exchanged, strategies discussed and new activities devised and refined. Post-testing occurred at the end of the course. Delayed testing occurred five months later.

The overall results

Multivariate analysis of variance (MANOVA) was used to analyse the three levels of test (ie: pre, post and delayed) and two types of class (ie: control and intervention). A covariate approach (using pre-test scores as covariates) was used to add power to the analysis by adjusting for the small inequalities which existed between groups at the start of the quasi-experiment. For simplicity, only the multivariate results are given here (Table 1) (but see Tanner & Jones, 1995 or Tanner, 1997 for further details).

The active metacognitive skills of planning, monitoring and evaluating:

Both the control and intervention classes improved over the period of the quasi-experiment although the intervention classes improved more than the control classes in the post-test and this improvement was sustained in the delayed-test. The effect size was small (0.19), but significant at the 0.1% level for the written tests (Table 1).

As the active metacognitive skills had been taught in practical mathematical modelling contexts, such ìnear transferîmight be considered unsurprising. Its achievement was non-trivial, however, as the pupils were required by the assessments to form their own problems within open situations, plan, identify and control variables, choose simple strategies, monitor their work, collect and organise their data, find relationships, evaluate and reflect on their results. These are the process skills of mathematics and are identified in the National Curriculum for England and Wales as worthy of learning in their own right.

Passive metacognitive knowledge or ìknowing what you knowî

The results here were not clear cut. Although the intervention classes improved in their forecasting more than the control classes in the post-tests and this improvement was sustained in delayed-testing, the effect of type of class was not significant at the 5% level. The postcasting of the intervention classes improved more than the control classes in the post-tests and this improvement was sustained in delayed-testing. This was significant beyond the 5% level but was limited to an extremely small effect size of 0.02 (Table 1).

Mathematical cognitive development

This showed a similar pattern to the active metacognitive skills but a smaller effect. The intervention classes improved more than the control classes in the post-test and the advantage was largely sustained at the delayed-test but the effect size (0.02) was extremely small (Table 1).

The content of the cognitive section of the written paper was not taught directly by the course, and intervention class teachers were careful to avoid practising questions like those on the test in advance. Given that the intervention classes had less teaching in the normal mathematics curriculum over the period of the quasi-experiment, it might have been expected that the control classes would generally have outperformed the intervention classes. This small overall effect is claimed, therefore, to be an example of mathematical thinking skills paying for themselves through far transfer.

Table 1: Multivariate tests of significance for the effect of type of class

Variable	Hotellings	F value	Hypoth DF	Error DF	Sig of F	Effect size
Metacog skill	.235	43.67	2	371	.000	.191
Forecast	.013	2.30	2	363	.101	.013
Postcast	.022	3.78	2	341	.024	.022
Cognitive dev	.021	3.89	2	369	.021	.021

The four teaching styles

Analysis of the qualitative data collected through participant observation led to the classification of the teachers into four characteristic groups according to the teaching styles employed. These were *taskers, rigid scaffolders, dynamic scaffolders* and *reflective scaffolders* (see Tanner, 1997 for detailed descriptions).

The *taskers* focused on the demands of the task rather than the targeted metacognitive skills. The *rigid scaffolders* focused on planning, but rather than helping pupils to develop their own plans, aimed to share their own plans with the class. The scaffolding support provided by their questioning constrained pupilsí thinking, leading them down a pre-determined path. These two groups of teachers were the least successful. The *taskersí* classes showed no advantage over their controls in any test. The *rigid scaffolders* showed an advantage in only the metacognitive delayed test with a very small effect size (0.09) significant at the 5% level. This paper focuses on the other two groups of teachers.

The dynamic scaffolders

The *dynamic scaffolders* made full use of the social structure of ìStart-stop-goî to frame their pupilsí behaviour and constrain them to act as experts rather than novices. This included the granting of significant autonomy to pupils, particularly in the early stages of planning. Their scaffolding was dynamic in character and was based on participation in a discourse in which differences in perspective were welcomed and encouraged. The most significant participant in the discourse was the teacher, who validated conjectures and used focusing questions to control its general direction ensuring that an acceptable whole class plan was generated. The participation framework was equivalent to ìlegitimate peripheral participationî within an apprenticeship model of learning (Lave &Wenger, 1991), the autonomy and responsibility of the pupils being limited by the teacher's desire to negotiate a plan to a pre-determined template. The discourse focused on both procedural knowledge and conceptual knowledge and during the planning and monitoring sessions, articulation and objectification of explanation was encouraged, making the explanation itself the object of the discourse. This was the only form of evaluation or reflection used by the *dynamic scaffolders*, however, and is characterised as ìreflection in actionî as opposed to ìreflection on actionî (Schˆn, 1990).

The *dynamic scaffolders* were very successful in accelerating the development of the active metacognitive skills of planning, monitoring and evaluating in the context of mathematical modelling, that is in near transfer, with a small to medium effect size (0.36) which was significant beyond the 0.1% level (Table 2). However they failed to achieve a significant advantage for their classes in either passive metacognitive self knowledge or ìfar transferî into the content areas of mathematics. It is conjectured that, although active metacognitive skills may be necessary in the learning of new knowledge, they are not sufficient. Metacognitive self knowledge may also be necessary for far transfer.

Table 2: Multivariate tests of significance for the effect of type of class for *dynamic scaffolders*

Variable	Hotellings	F value	Hypoth. DF	Error DF	Sig of F	Effect size
Metacog skill	0.550	21.27	2	77	.000	0.36
Forecast	0.040	1.52	2	76	.226	0.04
Postcast	0.015	0.53	2	70	.590	0.02
Cognitive dev	0.020	0.80	2	78	.453	0.02

The reflective scaffolders

The reflective scaffolders also used the social structure of ìStart-stop-goî to constrain their pupils to act as experts rather than novices, and dynamic scaffolding in a conjecturing atmosphere to lead the discourse in their classrooms. They granted their pupils more autonomy, however, encouraging several approaches to the problems rather than constraining the discourse to produce a class plan. Pupils thus had to evaluate their own plans in comparison with the other plans in the posing, planning and monitoring phases of the lessons. The participation framework had fewer of the characteristics of apprenticeship, with pupils drawing on the help of the expert and taking a greater responsibility for an end product of their own design rather than taking limited responsibility for an element in the design of a ìmasterî. The characteristic feature of the *reflective scaffolders*, however, was their focus on evaluation and reflection. During interim and final reporting back sessions, scientific argument was encouraged to make the explanation an object of the discourse. Peer and self assessment was encouraged through group presentations of draft reports before redrafting for assessment. They deliberately generated a reflective discourse (Cobb et al, 1997) after activities to encourage self evaluation and reflection on process. Collective reflection does not equal reflected abstraction, but it is conjectured that during collective reflection, opportunities arise for pupils to reflect on and objectify their previous actions as they engage in reflective discourse (Wheatley, 1991; Cobb et al, 1997).

The *reflective scaffolders* were very successful in accelerating the development of active metacognitive skills, achieving near transfer in practical modelling situations with a medium size of effect (0.4) which was significant beyond the 0.1% level of significance. They also succeeded in accelerating the development of passive metacognitive self knowledge in forecasting and postcasting. The effect sizes were very small (0.07 and 0.14), but significant beyond the 5% and 1% levels respectively (Table 3). They were the only group of classes to achieve this and it is conjectured that this was due to their emphasis on self evaluation and reflection.

The *reflective scaffolders* also succeeded in accelerating development in the content domains of mathematics measured by the cognitive test, again the only group of classes to achieve this far transfer. The size of effect was small (0.21), but was statistically significant beyond the 0.1% level and approximated to a year's development (Table 3).

Table 3: Multivariate tests for the effect of type of class for *reflective scaffolders*

Variable	Hotellings	F value	Hypoth. DF	Error DF	Sig of F	Effect size
Metacog skill	0.652	43.34	2	133	.000	0.40
Forecast	0.073	4.73	2	130	.010	0.07
Postcast	0.161	9.82	2	122	.000	0.14
Cognitive dev	0.272	18.09	2	133	.000	0.21

Discussion

The ìDynamic scaffoldersî were operating a model of cognitive apprenticeship which included authentic tasks, student autonomy and dynamic scaffolding but although they were very effective in teaching for near transfer, they failed to achieve far transfer. Individual construction was subordinated to the dynamics of the apprenticeship model, whereas the *reflective scaffolders* encouraged both cognitive apprenticeship and individual construction. The dynamics of ìStart-stop-goî were internalised through participation in social processes. Learning this procedural knowledge was achieved through an apprenticeship model, organised and controlled by the teacher. Through participating in a scientific discourse led by an expert using dynamic scaffolding, pupils learned to internalise the processes of scientific argument and ìargue with themselvesî. They also learned that mathematics ìmade senseî and that they could ìmake their own senseî of what occurred by making their own tentative conjectures and constructions and linking them with prior schemata.

Participation in reflective discourse encouraged reflective abstraction and the objectification of explanation. It is conjectured that the processes of mathematisation and problem solving, once objectified through individual construction, were thus knowledge rather than mere information and thus capable of being used elsewhere. Collective reflection provided both a social model and an opportunity for reflected abstraction.

The reflective discourse focused on the processes of mathematisation, abstraction and generalisation in the service of understanding structure. Participation in reflective discourse may have encouraged the development of a mathematical disposition or point of view (Schoenfeld, 1994; Cobb et al, 1997) which might be expected to be transferable.

The active metacognitive skills of planning, monitoring and evaluating are generic rather than specific and these skills can survive near transfer to similar modelling contexts. It is conjectured that participation in reflective discourse can encourage objectification and the development of metacognitive self knowledge thus enhancing the transfer potential of such skills. It is further conjectured that the combination of active metacognitive skills and passive metacognitive knowledge supports both the application of old mathematics to new contexts and the learning of new mathematics.

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A computer environment to encourage versatile understanding of algebraic equations

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Learning algebra is very difficult for many students, and one of the major obstacles identified in the research literature is a 'cognitive gap' presented by equations of the form ax+b=cx+d, which require operation on the variable for solution. In this small-scale, preliminary study we assessed the value of a computer environment which enables students gradually to investigate algebraic expressions and equations. The initial results are very encouraging and suggest that this may be a way to assist students to learn algebra in a versatile way.

Background

There is a wide range of research informing us that children find great difficulty in understanding of what has been called the algebra of generalised arithmetic (e.g. Küchemann, 1981; Wagner, Rachlin & Jensen, 1984; MacGregor & Stacey, 1997). One of the problems with understanding how to solve linear equations is that, prior to the introduction of algebra, students become accustomed to working in an arithmetic environment where they solve problems by producing a numerical 'answer', leading to the expectation that the same will be true for algebra. Thus, for many the '=' sign is an indication that an answer is to be expected (Kieran, 1981), leading them to implement arithmetic or informal methods in the solution of algebraic equations. However, when they are faced with an equation of the form $ax+b=cx+d$, which requires one to operate on the unknown, they often face an insurmountable obstacle. Herscovics and Linchevski (1994) have described this inability of students to operate with or on the unknown, as characterising a demarcation between arithmetic and algebra, what they term a *cognitive gap*, based on the *didactic* cut of Filloy and Rojano (1984), and they concluded that many students were reduced to performing meaningless operations on symbols they did not understand. The question is, how can one improve conceptual understanding of equation in a way which addresses obstacles like this? Linchevski and Herscovics (1996) report on their teaching experiment using decomposition and cancellation of identical terms for solving linear equations, and while they experienced a measure of success, they found "that some of these obstacles are rather robust and perhaps should not be dealt with incidentally. . . " (*ibid*, p. 39). Pirie and Martin (1997, p. 141) believe that the obstacle results from teaching algebra "through appeal to arithmetic parallel thinking" and they also describe a method resulting in some success, using a method based on grouping singleton boxes as multiples of variables.

We believe that an important part of the learning process lies in recognising that mathematical objects have two distinct but complementary faces, namely as processes or objects, or in the words of Sfard (1991), they possess operational and structural aspects. Dubinsky & Lewin (1986) relate these by discussing how learners *encapsulate* processes as objects. In a later revision of this, designated APOS theory, Cottrill *et al.* (1996) discuss how actions on objects become interiorised as processes, which in turn may be encapsulated as objects. This latter stage occurs when "the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations." (*ibid*, p. 171). Concepts may then be viewed as schematic constructions comprising actions, processes and objects. Sadly, many students fail to encapsulate algebraic processes as objects and so are left with a *process-oriented* (Thomas, 1994; Kota & Thomas, 1998) view of algebra rather than progressing to the point where they can think in a *versatile* manner (Tall & Thomas, 1991; Hong & Thomas, 1998). This *versatile* perspective is one where, through encapsulation, learners attain a global view of the objects and/or processes comprising a concept, alongside its components, or constituent process(es), which they can think sequentially about, and inter-relate these two perspectives. In contrast, thinking primarily in terms of mathematical processes structures thinking towards procedural methods, algorithms and presentations (Thomas, 1994).

Attaining a versatile, conceptual understanding of algebra involves learning its concepts, which are primarily (at least initially) variable, expression, function and equation, via actions which enable one to construct a view of them as process or object. Applying this to equations, versatile learning requires students to be able to investigate processes of building and solving equations, but at the same time to build a global view of equation which sees it as an object based on the equivalence of two functions or expressions. This property of equivalence is necessary to underpin the processes that students engage in, such as adding the same terms to both sides and/or cancelling equivalent terms. Hence, equations may be categorised as *procepts* (Gray & Tall, 1991, 1994), a combination of mathematical symbols, a process and an object, and student thinking should be directed toward activities which build this perspective.

This research sought to investigate whether it is possible to improve students' proceptual understanding of equations by giving them an environment in which they could manipulate examples, predict and test results, and gain experiences on which a versatile understanding of expression and equation as process and object could be built.

Method

This was very much a preliminary, small scale study involving case studies of a small number of students. While pre- and post-tests were used, the main aim was to investigate individual student's improvements in understanding. The second named author was the class teacher of all the students and conducted the tests and interviews.

Subjects

Twelve students of similar ability on standardised testing (TOSCA Stanines of 7 to 9+) were chosen; six from year 7 (age 11 years) and six from year 8 (age 12 years), with three females and three males in each group. This selection took place in May 1997, allowing for the year 7 children to be well settled into their school year. The year 7 students had not been taught any algebra at all, while the year 8 students had been introduced to algebra the previous year. In the event, of the initial twelve students selected to participate, one of the year 7 group was unavailable to participate in the pre-test and so was removed from the study group, reducing it to eleven students.

Procedure

Initially, all the students were briefly interviewed individually about their understanding of algebra, and after their interview they were given a pre-test of 40 linear equations to solve, based on questions from a test used by Herscovics and Linchevski (1994). In this test the linear equations were grouped into 8 sections, increasing in difficulty, beginning with $14 + n = 43$ and ending with $5n + 12 = 3n + 24$, an equation spanning the cognitive gap. The comparison with Herscovics and Linchevski (1994) was useful because their research highlighted both the relative difficulty of the questions and the types of solution procedures used by students. All the students were allowed to use a calculator when completing the tests.

Each student was supplied with their own disk copy of a computer software program, entitled *Dynamic Algebra,* comprising an environment for investigating algebraic concepts, covering the concepts of expression and equation in algebra, and this was used as a teaching tool. They were initially shown how to use the program and were then instructed to trial it on their own, with any queries being followed up on an individual basis. The students then spent the next six weeks using the program and working through its various levels. Since they all had access to a computer at home, they were also encouraged to use the program for at least fifteen minutes per night. The children also participated in their daily classroom mathematics programme, and at the end of the six weeks they were given a post-test comprising the same test used for the pre-test.

The Dynamic Algebra computer environment

The *Dynamic Algebra* program comprises three basic areas: substitution of a variable into an expression; equivalence of expressions; and equations. Each of these areas is further broken down into three categories: machine choice; abstract choice and a test situation to check understanding of the particular area. The program is built on the concept of variable as a named store, one which had worked well in other contexts (Tall & Thomas, 1991; Graham & Thomas, 1997), and students see a variable label under a box containing its current value. Figure 1 shows a sample screen from the trial and error section on equation solving. This section of the program is extremely important since it combines the above model of variable with the idea of substitution in an expression (or function value) and equivalence of both sides of an equation. Students investigate solutions by entering values for the variable, here *u*, until both sides are seen to be equal in value, that is the difference between the two sides is zero.

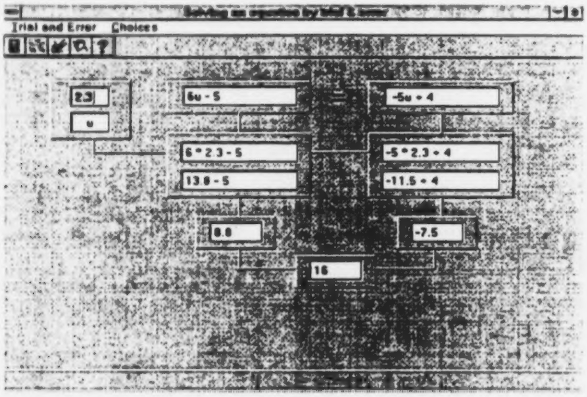

Figure 1: A screen showing a trial and error approach to solving an equation

All the boxes in this mode other than the variable's value are completed by the program, which shows both the full expressions with substitution of the variable's value, and the simplified versions. Although the equations are generated automatically, the difficulty level of the linear function used may be chosen, and there is the option of user entry of expressions and equations.

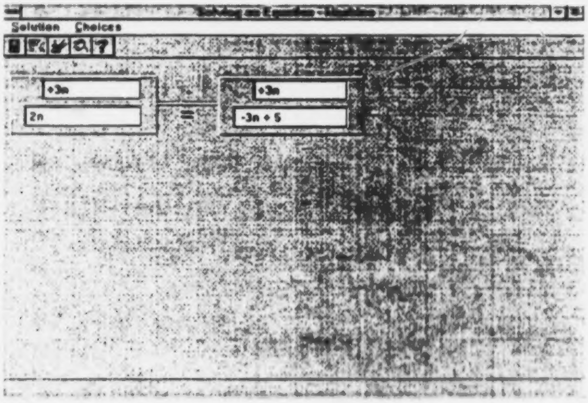

Figure 2: A solution to an equation involving the same operation on both sides

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Figure 2 shows a sample from the section on solving equations by adding like terms to both sides. At any point the program constrains the user to entry of a 'correct' multiple of the variable or a constant. Should a student be unable to continue at any particular point in a solution then they may click on the *coach* button and get the assistance of an imaginary coach who fills in the next box. A 'smiley face' greets a correct solution (see figure 3).

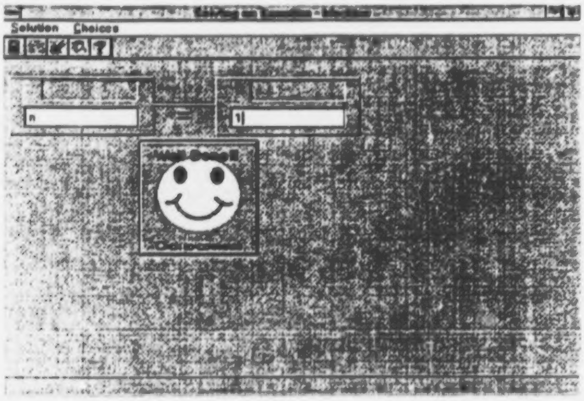

Figure 3: A screen showing a successful solution to an equation

In this way the students were able to work through many of the areas of the computer environment on their own.

Results

Year 7 students

In their preliminary interview the students were asked what they knew about $3n$ to gauge their knowledge of algebra. The year 7 students, who were understood not to have studied any algebra, were naturally hesitant in their responses. One student thought that it could be 3 x 3, but most had no idea, confirming that they had no prior knowledge of algebra. To confirm their responses the students were asked to replace the n with a 2 or a 5, and the responses were similar to the previous question, with the students responding 32 or 35, substituting the number without considering that any multiplication was required. One might expect this lack of understanding to be a severe handicap in solving equations, but on the pre-test, these students, coded A to E, correctly solved 22, 29, 10, 31 and 24 of the 40 questions respectively. Three of the students B, D and E, were even able to solve some equations such as:

$$n + n = 76, \quad 4n + 17 = 65 \quad \text{and} \quad 5n + n = 78$$

even though they had been unable to attach the correct meaning to a multiple of n. For the first of these questions, this phenomenon was noticed by Linchevski and Hescovics (1996, p. 43) who commented that for $n + n = 76$ etc.

> Most students immediately divided 76 or 50 by 2. These results indicate that when the terms in the unknown are singletons, i.e. without any coefficient, the majority of students have a natural tendency to mentally group the terms in the solution process.

However, it is very surprising that this could be extended to the second and third questions, which appear to require understanding what the $5n$ means. It would seem that these students used a combination of trial and error substitution, invented and informal procedures, possibly adding the coefficients and dividing by the total in some questions, correctly emulating algebraic procedures. Certainly, student B knew what $4n$ meant by the time of the pre-test, and was also using trial and error substitution, since, for the question $3n + 5n + 4n = 19$, he tried $n = 1.5$ and wrote $4.5 + 7.5 + 6 = 18$, then tried 1.6 and got 19.2 and then 1.55 getting 18.6, so as his answer he wrote, 'More than 1.55, less than 2', since he was concerned that if he wrote 'between 1.55 and 1.6' this would not be

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considered as an answer. He clearly did not use the calculator to find 19/12. Students A and C, as one would expect, were unable to answer correctly any question involving a multiple of n, or more than a single variable in a question. For the final three questions, each of which had an unknown on both sides of the equation, there was only one correct answer, from student B, and it turned out, on investigation, that this was a fortuitous guess.

After their work on the computer, as table 1 shows, their marks had improved significantly to 40, 38, 39, 40 and 35 respectively.

Table 1: An analysis of the pre- and post-test mean scores of the year 7 and year 8 students

Year	Pre-Test	Post-Test	t	p
Year 7	23.2	38.4	3.97	<0.05
Year 8	31.5	38.8	3.71	<0.05
Combined	27.7	38.6	4.76	<0.001

Student A, who improved from 22 to 40, and who could not deal with any equation other than those with a single n, not only got all the questions correct but had built an understanding of the need to treat each side of an equation the same. For the equation $n + 15 = 4n$, he was able to write $3n = 15$, $n = 5$, immediately, but for the final question: $5n + 12 = 3n + 24$, he wrote:

$$2n + 12 - 12 = 24 - 12; \qquad 2n = 12; \qquad n = 6$$

and when asked afterward what he had done he dictated the 'missing' first line.

$$5n - 3n + 12 = 3n - 3n + 24.$$

This remarkable improvement, to a point where he had avoided the cognitive gap, was emulated by student C, who had the lowest pre-test score of 10, and who improved to 39. For the equation, $n + 15 = 4n$, she wrote:

$$-n + n + 15 = 4n - n; \quad 15 = 3n; \qquad \text{and the answer as 5,}$$

with no concern about the $-n$ appearing first. Unfortunately she was somewhat too rushed to finish the final question, but she did enough to show that she too had constructed the principle of treating both sides the same, writing,

$$5n - 3n + 12 = 3n - 3n + 24$$

$$2n + 12 = 24$$

$$2n + 12 - 12 =$$

$$2n = 44; \qquad n = 22$$

The blank in the third from last line seems to be an indication that she had made a mistake rather than having an error in her understanding.

Two limitations of the test instrument which emerged from this study were: the questions were not difficult enough at the post-test, in particular there should have been more questions with variables on both sides of the equation; and students were not required to show their working. A consequence of these two limitations was that most wrote down only the answers, or if they did show working they erased it afterward. This last point was true of all three students, B, D and E. However, student B improved on the question $3n + 5n + 4n = 19$ so that he was able to write 1.58 as the solution, probably by adding the ns and dividing 19 by 12. He also wrote some indication that he may have been performing the same operation on both sides when he wrote, $5n - 3n + 12$ as partial working on the final question, $5n + 12 = 3n + 24$ (note that they had not been exposed to 'change side, change sign' in any form). He commented that "The program was easy to use and fun." Student D got all the questions correct, but with no indication of her methods , and the same was true of student E. however she thought that it was "a good idea to have a programme on the computer. It's good because you can choose the level which is right for you. Overall I thought it was really good."

Certainly, from a base of no knowledge at all of algebra we can say that the level of understanding and ability to solve the equations at the end of the six week period on the computer was very pleasing.

Year 8 students

Since the year 8 students, coded F to K, had already been exposed to an introduction to algebra, when asked in their preliminary interview what they knew about $3n$, four of the six students replied that it was the same as saying $3 \times n$. One student, H, was not sure and another suggested that it could be 3×3. On substituting 2 and 5, most replied that it was $3 \times 2 = 6$, or $3 \times 5 = 15$, although H did respond that it was 32 or 35.

These students, who scored 38, 29, 27, 26, 37 and 32 respectively on their pre-test appeared to be too good to gain much from their use of the computer environment. However, as table 1 shows, their mean score did improve significantly, and we also noted that the methods they used improved considerably. Student F used trial and error and checking by substitution on the pre-test and although she still used this method on some questions on the post-test, when she scored 39, for the equation: $3n + 5n + 4n = 19$, she wrote:

$$12n = 19, \qquad n = 19/12,$$

this being a question she did not attempt on the pre-test. She also described the program as "quite helpful". Student I, also wrote that the program "really improved my understanding of algebra. . . They should keep this program". He got 39 correct, making only a calculator entry mistake on an early question. Student J, who managed 37 correct on the pre-test, was, even then able to solve $n + 15 = 4n$, by writing:

$$15 = 3n; \qquad n = 5,$$

although for the final question he asked: "Is this saying that $5n + 12$ is the same as $3n + 24$?", and was clearly bemused by it and unable to make any attempt. On the post-test he scored 39, again making only a calculator entry mistake on an early question but this time correctly solving $5n + 12 = 3n + 24$, although with no indication of method. However, his comments on the program included the encouraging remark: "Helped you learn formulas eg What you do to one side you do to the otherside [sic]."

The remaining three students, G, H and K showed the best improvements in understanding. G used guess and substitution on all her questions on the pre-test, getting 29 correct, but commented on questions with more than one n, when interviewed afterward, "Does n have a variety of numbers or are they the same?"

After the computer work she appeared to have no problem with this and for the final question, $5n + 12 = 3n + 24$ wrote:

$$5n - 3n + 12 = \cancel{3n} - \cancel{3n} + 24; \quad 2n + 12 = 24; \quad 2n + \cancel{12} - \cancel{12} = 24 - 12;$$

$$2n = 12; \qquad n = 6.$$

This method and the 'cancelling' of the $3n$ and the 12 are significant steps forward. G commented on her computer work: "I quite liked simplification. Algebra is easier on the computer than on paper. . . I worked by myself how some of these equations as Mrs Hall did not show me." H, with 27 on the pre-test, used guess and substitute, and could not solve any equation with a multiple of n in it. On the post-test she got 38 correct, including all the final three questions, for one of which she wrote:

$$4n + 9 = 7n; \qquad \cancel{4n} - \cancel{4n} + 9 = 7n - 4n; \qquad 9 = 3n; \qquad n = 3.$$

cancelling the 4*n*s at stage one. Of the algebra program she said "If I didn't know something then the answer guy was a big help. . . The way it took you through step by step was good. Overall I think the algebra program was good." Finally, student K, on the pre-test scored 32 and was able to add like terms and use inverse operations in his solutions, for example, on $3n + 4n = 35$ he used 35/7. He was not able to answer the final question. After the computer work he got everything correct and worked through the final question exactly the same as G above, but without the physical cancelling of terms. Again, these gains are worthy of note, and lead us to believe that the program has value in helping students build understanding in algebra.

Not everything about the computer environment was to the liking of the students. They wanted to see some improvements and these included: the need for an in-built calculator; "you should be able to get the answers wrong in the tests"; not enough instructions; the 'smiley face' was annoying.

Conclusion

As we have made clear, this was a preliminary study on the use of the *Dynamic Algebra* environment. It was pleasing to note that, not only did the students as groups perform significantly better in terms of the overall number of questions they got correct, but they had developed a more versatile view of equation. This was shown by the clear evidence that after the computer work 6 of the 11 students were known to be using a method of solution where they applied the same operation to both sides of the equation and 'cancelled' terms, even operating on the variable and 8 of the 11 were able to solve the equation $5n + 12 = 3n + 24$. It appears that they had successfully avoided the cognitive gap described by Herscovics and Linchevski (1994). The students found the programme interesting and stimulating, but also pointed out improvements, including the need for written instructions (the program has a built-in Help menu). These results have been very encouraging and we are now designing a larger study to confirm them, and to take into account the lessons learned.

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Teacher Beliefs About the Learning and Teaching of Mathematics: Some Comparisons

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During the last four years, a survey instrument was administered to 603 primary and 336 secondary teachers in both government and Catholic schools across an urban and a rural school region in New South Wales. This paper reports on comparisons concerning the espoused beliefs about mathematics, mathematics learning and mathematics teaching of the teacher respondents and demonstrates significant differences between the teachers across the regions, types of school and gender.

Introduction

This report continues the research agenda presented at MERGA 17 (Perry & Howard, 1994) which set the basis for investigations of primary and secondary teachers' use of manipulatives (Howard, Perry & Conroy, 1995; Howard, Perry & Conroy, 1996; Howard & Perry, 1997; Howard, Perry & Tracey, 1997) and their espoused beliefs about mathematics, mathematics learning and mathematics teaching (Perry, Howard & Conroy, 1996; Howard, Perry & Lindsay, 1997). Evidence suggests that such beliefs play a major role in determining how teachers teach (Barnett & Sather, 1992; Pajares, 1992; Weissglass, 1992). The focus of this paper is the comparison of the espoused beliefs of primary and secondary mathematics teachers across identified demographic variables.

Teacher beliefs about mathematics, mathematics learning and mathematics teaching

The development of beliefs about the nature of mathematics and how mathematics is done "*are important not only because they influence how one thinks about, approaches, and follows through on mathematical tasks but also because they influence how one studies mathematics and how and when one attends to mathematics instruction*" (Garofalo 1989, p. 502). Critical to the classroom implementation of the learning and teaching of mathematics is the teacher and, in particular, the beliefs of the teacher. All teachers of mathematics hold beliefs about mathematics, mathematics learning and mathematics teaching. These beliefs influence and guide teachers in their decision making and implementation of teaching strategies (Baroody, 1987)

Teachers' espoused beliefs about mathematics, mathematics learning and mathematics teaching fall along a continuum from a traditional teacher-centred point of view where the teacher is seen to be the deliverer of knowledge and skills ("transmission") to a view where children play a central role in constructing their own mathematics ("child-centred") (Anderson, 1996; Cobb, 1988; Yackel & Cobb, 1996; Kuhs & Ball, 1986). Such a continuum has been used by the authors to profile the beliefs held by primary school teachers (Perry, Howard & Conroy, 1996) and secondary teachers (Howard, Perry & Lindsay, 1997).

Methodology

The data for this investigation were collected using a specifically designed questionnaire covering the following areas:

1. subject demographics such as gender, age, position in school, nature of teacher training, length of teaching experience, class(es) currently taught, class size, classes taught over the last ten years;
2. use of manipulatives in mathematics learning and teaching such as which are used, why and how they are used, and the areas of mathematics in which they are used;
3. beliefs about mathematics, mathematics learning and mathematics teaching.

The questionnaire relied on the self reporting of the teachers, paralleling much of the work reported in Hatfield (1994), and was administered over 1995, 1996 and 1997 to the samples described in Table 1. All schools involved in the study were located in either the south western suburbs of Sydney or the north coast of New South Wales.

Year	School level	School type	Number of Schools	Responses received
1995	Primary	Government	25	252
1996	Secondary	Government	37	198
1996	Secondary	Catholic	15	51
1997	Primary	Government	114	244
1997	Primary	Catholic	32	107
1997	Secondary	Government	23	44
1997	Secondary	Catholic	12	43

Table 1 Number of schools surveyed and number of responses

In all cases, the questionnaire was posted, with reply paid envelopes, to the Principals of the schools after first making telephone contact to obtain their initial approval to undertake the survey in the schools and to ascertain the number of primary teachers or secondary mathematics teachers in each school. In the secondary schools, telephone contact was also made with the Head Teacher (Mathematics).

Data from these responses were analysed using the SPSS-X program to provide descriptive statistics for the demographic data and the use of manipulatives and comparative statistics regarding teacher beliefs about mathematics, mathematics learning and mathematics teaching.

For this paper, a confirmatory factor analysis was employed to confirm the authors' a priori predictions of the clustering of the beliefs items used in the questionnaire related to the continuum (transmission to child-centredness) of teachers' mathematical beliefs. Analysis of variance was used to investigate the data in relation to these two factors across the demographic variables.

Results

Confirmatory factor analysis

To gather data concerning teachers' espoused beliefs about mathematics, mathematics teaching and mathematics learning, an instrument consisting of twenty statements was designed following an extensive literature review (Perry, Howard &

Conroy, 1996) (see Table 2). A three-point Likert response scale - 1 (disagree), 2 (undecided) and 3 (agree) - was used with these statements. Hence, higher scores indicate more adherence to the particular belief represented by the statement.

Mathematics

▲ 1. Mathematics is computation
▲ 2. Mathematics problems given to students should be quickly solvable in a few steps
● 3. Mathematics is the dynamic searching for order and pattern in the learner's environment
4. Mathematics is no more sequential a subject than any other
● 5. Mathematics is a beautiful, creative and useful human endeavour that is both a way of knowing and a way of thinking
▲ 6. Right answers are much more important in mathematics than the ways in which you get them

Mathematics learning

● 7. Mathematics knowledge is the result of the learner interpreting and organising the information gained from experiences
● 8. Students are rational decision makers capable of determining for themselves what is right and wrong
▲ 9. Mathematics learning is being able to get the right answers quickly
● 10. Periods of uncertainty, conflict, confusion, surprise are a significant part of the mathematics learning process
● 11. Young students are capable of much higher levels of mathematical thought than has been suggested traditionally
▲ 12. Being able to memorise facts is critical in mathematics learning
● 13. Mathematics learning is enhanced by activities which build upon and respect students' experiences
● 14. Mathematics learning is enhanced by challenge within a supportive environment

Mathematics teaching

● 15. Teachers should provide instructional activities which result in problematic situations for learners
▲ 16. Teachers or the textbook - not the student - are the authorities for what is right or wrong
▲ 17. The role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge
● 18. Teachers should recognise that what seem like errors and confusions from an adult point of view are students' expressions of their current understanding
● 19. Teachers should negotiate social norms with the students in order to develop a cooperative learning environment in which students can construct their knowledge
20. It is unnecessary, even damaging, for teachers to tell students if their answers are correct or incorrect

Table 2 Beliefs statements
▲ denotes that the item loads significantly on "transmission"
● denotes that the item loads significantly on "child-centredness"

A priori predictions were made about how these statements would cluster around the factors of transmission and child-centredness. A confirmatory factor analysis, using principal axis factoring and an oblique rotation solution method, was conducted. Using a criterion of > .30 (or < - .30) for the significance of a factor loading, it was found that the 7 items marked ▲ in Table 2 loaded significantly onto one factor - "transmission" and the 11 items marked ● in Table 2 loaded significantly onto the other factor - "child-centredness". Items 4 and 20 did not load significantly on either factor. This analysis confirmed the a priori predictions of the pattern of clusters among the items, hence, providing construct validation for the measure.

Demographic data

There were 939 respondents - 603 primary and 336 secondary teachers. Data concerning the location and types of schools in which the respondents taught are given in Tables 3 and 4. Table 5 shows the gender composition of the sample.

School level / Sector	Catholic	Government
Primary	107 (18%)	496 (82%)
Secondary	94 (28%)	242 (72%)

Table 3 Composition of sample by sector (N = 939)

School level / Location	North Coast	South Western Sydney
Primary	351 (58%)	252 (42%)
Secondary	87 (26%)	249 (74%)

Table 4 Composition of sample by location (N = 939)

School level	Female	Male
Primary	458 (76%)	143 (24%)
Secondary	138 (41%)	198 (59%)

Table 5 Composition of sample by gender (N = 939)

The majority of respondents had 6 or more years of teaching experience, as shown in Table 6.

School level	< 1 year	1 - 5 years	6 - 10 years	11 - 20 years	> 20 years
Primary	13 (2%)	89 (15%)	105 (17%)	199 (33%)	197 (33%)
Secondary	9 (3%)	45 (13%)	57 (17%)	116 (35%)	108 (32%)

Table 6 Teaching experience (N = 939)

The school positions held by the respondents in both primary and secondary schools are reported in Table 7.

School level	Principal / DP / AP	Other executive / Head Teacher (Maths)	Classroom teachers	Other teachers
Primary	120 (20%)	54 (9%)	366 (61%)	59 (10%)
Secondary	10 (3%)	67 (20%)	247 (74%)	11 (3%)

Table 7 Positions held in primary and secondary schools (N = 939)

The majority of teachers in the sample reported that they are qualified at the 3 or 4 year trained level. Some have undertaken postgraduate study and some identified themselves as 2 year trained. Details are given in Table 8.

School level	2 years	3 years	4 years	> 4 years
Primary	53 (9%)	245 (41%)	280 (46%)	24 (4%)
Secondary	3 (1%)	21 (6%)	281 (84%)	29 (9%)

Table 8 Years of teacher education (N = 939)

Analysis of variance in beliefs data

The data were tested for significant levels of difference on the two confirmed factors (transmission and child-centredness) across the demographic variables. In this paper, comparison of beliefs data for the variables of school sector, school location, school level and gender of respondents is reported.

Sector: Catholic school respondents (mean = 0.3219) scored significantly higher ($t=-2.69$, $p<.01$) on the child-centredness factor than respondents from government schools (mean = 0.1194) but there was no significant difference between the two sectors on the transmission factor.

Location: North coast school respondents (mean = 0.2077) were more likely to support statements relating to child-centredness ($t=-2.10$, $p<.05$) than respondents from south western Sydney schools (mean = 0.0834) but there was no significant difference between the two locations on the transmission factor.

Level: Primary school teachers who identified themselves as "mainly K-2" (mean = 0.2047) or "mainly 3-6" (mean = 0.1695) or "mainly 5-6" (mean = 0.0714) reported significantly higher agreement with the child-centredness statements ($F=13.6738$, $p<.0001$) than did secondary mathematics teachers (mean = -0.2615). However, there were no significant differences on child-centredness amongst these three groups of primary teachers.

On the transmission factor, no significant differences were found between the secondary mathematics teachers, "mainly K-2", mainly 3-4" and "mainly 5-6" teachers. However, although not significant, there was a discernible trend towards secondary teachers scoring higher on this factor than primary teachers.

Gender: Female teachers (mean = 0.1006) scored significantly higher ($t=4.42$, $p<.001$) on the child-centredness factor than did male teachers (mean = -0.1667). Male teachers (mean = 0.0691) scored significantly higher ($t=-2.31$, $p<.05$) on the transmission factor than did female teachers (mean = -0.0629).

Discussion

Justification has been given through the use of confirmatory factor analysis for the existence of two factors - transmission and child-centredness - to be used in categorising teachers' espoused beliefs about mathematics, mathematics learning and mathematics teaching. The survey used in this study is one means by which data can be gathered to profile teachers on these two factors.

The survey findings give rise to a number of questions which need further investigation. Some possible answers to these questions are given below, based on the authors' experience and reading. The authors intend to test these through focus group interviews during 1998.

Why do Catholic school teachers score higher on child-centredness than government school teachers? Catholic schools emphasise their pastoral care role and concern for the student as an individual. Thus, Catholic school teachers may have a more focussed concern for the child in all their teaching, including mathematics, than government school teachers.

Why is there a higher level of child-centredness among north coast teachers than those teaching in south western Sydney? Teachers in south western Sydney may be more concerned with management issues than north coast teachers because of the perceived classroom pressures associated with this area of Sydney and the relatively less-experienced group of teachers in this area.

Another possible reason arises from the relative socio-economic levels of each of the areas. There is ample evidence that socio-economic status is highly correlated with students' achievement. Haberman (1994) suggests that the local social context leads to a preference for the maintenance of the status quo rather than change. The status quo in many urban schools in low socio-economic areas is what Haberman (1994, p.17) refers to as "the pedagogy of poverty". Hatton (1994) describes this pedagogy as "a highly directive style of teaching based on rote learning of the basics, formulated without reference to adequate pedagogic or social theory and typically implemented in tough urban schools" (p.15). Haberman (1994) suggests that "[T]he pedagogy of poverty requires that teachers who begin their careers intending to be helpers, models, guides, stimulators, and caring sources of encouragement transform themselves into directive authoritarians in order to function in urban schools" (p.19).

Why do primary teachers score higher on child-centredness than secondary mathematics teachers? The teacher education undertaken by primary teachers normally differs in focus to that of secondary mathematics teachers. Primary teacher education programs focus on children's learning more holistically than secondary mathematics teacher education programs with their main focus on content.

Primary school teachers normally interact with their class all day, they are able to get to know their children on an individual basis and they are not constrained by timetable matters as much as secondary mathematics teachers. Secondary teachers may well work under greater pressures of time and content coverage and certainly under greater examination pressures, although recent changes to basic skills testing have diminished this difference. Compared to primary teachers, it may be that secondary mathematics teachers just do not have the time to be as child-centred in their teaching.

Why is there no significant difference on child-centredness amongst the responses of the three categories of primary school teachers? In primary schools, there is usually a common structure across the seven years K-6, there is a common mathematics syllabus and there have been substantial efforts to create K-6 schools. Further, the K-6 mathematics syllabus is used extensively across primary schools and emphasises child-centred approaches to the learning and teaching of mathematics. (NSW Department of Education, 1989, pp 4-5).

Why do female teachers score significantly higher on child-centredness than male teachers? One possible reason is that women are perceived to be more anxious towards mathematics than men (Forgasz & Leder, 1996). Due to their personal appreciation of this anxiety, female teachers may be more child-centred in their teaching because they are more aware of the need to address the affective concerns of children. There are more female teachers in this sample teaching in the lower years of the primary school where the teaching/learning strategies used are more child centred than those in the later years of schooling.

Why do male teachers score significantly higher on transmission than female teachers? Male teachers may well base their teaching on their past experiences in classes where power relationships and the delivery of information -transmission of information- was the preferred mode of delivery by their teachers.

Why are there no significant differences across school level, sector or location on the transmission factor? It may be that teachers who tend towards a transmission approach are more satisfied with it because it has brought success in the past to both them and their students. They believe there is little reason for changing.

Conclusion

The survey used in this study is a valid instrument for the investigation of teacher beliefs about mathematics, mathematics learning and mathematics teaching. It could be used to help teachers, both pre-service and in-service, to know their own teaching styles.

The introduction of social constructs such as the 'pedagogy of poverty' require mathematics educators to question the relevance of particular approaches to mathematics learning and teaching which have been developed in particular social contexts. A variety of teaching approaches may be needed to meet the requirements of the diverse group of students in our schools. The significant differences found in this study point to the need for further careful analysis of the learning and teaching of mathematics in classrooms situated in differing social contexts.

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Exploring the Impact of CAS in Early Algebra.

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This paper reports on an investigation into the impact of the availability of a hand held computer algebra system (CAS) on student performance and patterns of algebraic thinking. In particular, their performance in a range of symbol manipulation tasks in the early stages of the middle school algebra curriculum is discussed.

Introduction

> *'Every cognitive technology, connected to mathematics in some implicit or explicit way, inevitably influences the practice of mathematics education even if this was not its initial aim (eg language, print, pens and pencils, computers).* Pea (1986)

The technology currently available on $250 hand-held symbolic calculators will perform almost all of the tortuous algebraic manipulations found in the secondary school maths curricula in all Australian states. Frequently termed a Computer Algebra System (CAS), this 'cognitive technology' raises questions about the mathematical experiences we believe our students should have. For example, what symbolic manipulations should our students be able to complete by hand?

To illustrate the immediacy of such questions, consider the following situation, which arose last year in a Year 12 mathematics class in Victoria. A teacher refused to authenticate a student's work in a major Year 12 mathematics assessment task, because the student had used a CAS to do algebra manipulations which he could not do himself (Roberts, 1997a). In a follow up to the original article (Roberts, 1997b), the teacher added the following comments:

> *'we did give him credit for correctly formulating the equations - he was intellectually very gifted, but was also very lazy, which was partly why his algebra was weak (he didn't practice, and was very careless/untidy)'*

Kutzler (1997) contends that when students perform an algebraic procedure such as equation solving, it is possible to distinguish two types of steps in the solution process: *choosing* - deciding the operation to be performed and *applying* - carrying out that operation.

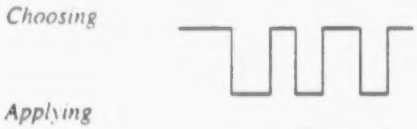

Figure 1

Kutzler argues that in most algebraic procedures, the higher level task of 'choosing' is repeatedly interrupted by the lower level task of 'applying' (see Figure 1). He suggests that a CAS allows the student to concentrate on the higher level tasks of choosing and responding to the feedback given by each choice.

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So should the work of Roberts' student have been authenticated? It appears that he may have known what steps needed to be performed, but could not complete them without the aid of CAS. In response to Roberts' article, Stacey suggested a newer set of criteria for assessing a students 'symbol sense' (Stacey, 1997). Students might be said to understand a procedure if:

a) they know what the procedure is for;
b) they know what sort of input is required;
c) they know under what conditions the procedure can be used;
d) they can make sense of the answer that the software produces;
e) they have some capacity to detect errors in the answers given by the software (sufficient to guard against input errors, for example).

These criteria attempt to provide checks that the student and not the technology is at the helm of a symbolic procedure. It is critical to identify, in a technology rich environment, which cognitive tasks should be retained by the individual and which tasks should be 'distributed' to the technology.

Determining the importance of 'symbol sense' may be guided by what has happened with the impact of the four-function calculator on arithmetic. We realised early that it made sense not to teach some things any more. Long division especially with large numbers and the use of logarithm tables and their many complicated variations left the mainstream curriculum. We also soon came to see that children could not use calculators sensibly if they did not have a good sense of number — button pushing is prone to silly mistakes, so knowing roughly what the answer will be like is essential (Groves, 1994). We now need to discover what experiences our students require in order to develop solid symbol sense, and how to create an environment where CAS may be both accessible and affordable for the majority.

Very little research has been conducted into the impact of CAS in the secondary school curriculum, and even less at the middle grades. The main thrust of the studies that have been undertaken is on the effects of replacing or de-emphasising by-hand symbol manipulations with computer generated symbol manipulations in post-secondary 'reform' calculus courses. Keller and Russell (1997) reports that five of seven semester long investigations found improved understanding of calculus concepts with no loss of computational skill. The other two studies found no difference in performance on departmental final exams. In a study incorporating CAS use in a quadratic functions unit with 14-15 year old students, Hunter commented that a CAS can be of benefit for the student's learning of abstract elements of algebra as long as the students were mathematically ready to use it (Hunter et al. 1995).

How might the classroom environment change in a CAS enriched environment? Clearly, a CAS can be used to perform and check by-hand calculations, and do this quicker and more effectively than we have been able to do in the past. In this sense, we are using the technology to 'amplify' the current curriculum. Alternatively, by holding up a mirror to current classroom practise, CAS may also act as a 'reorganiser' in the curriculum. Dorfler (in Hillel, 1993) argues that technology changes the nature of an activity or task in which it is employed in the following ways:

- stronger differentiation between planning & execution of mathematical algorithms/operations;
- requires more detailed planning/anticipation of results;
- change in nature of objects to operate on (eg complicated functions);

- change in representation of objects and processes (eg notation changes, functional representations, geometric objects);
- changes to sequencing of topics.

This study attempts to explore these issues in the context of the algebra curriculum.

Methodology

A teaching experiment was conducted to explore the effects of student use of a hand-held computer algebra system (a Texas Instruments TI-92) on their understandings of key algebra concepts and procedures related to these tasks. The tasks focussed on:

- symbol manipulation techniques;
- formulation of expressions and equations from word statements;
- solving problems which involve both formulation and symbol manipulation techniques.

The experiment was conducted in one single sex school and involved two classes of female year 9 students, taught by two teachers. One class was assigned as the experimental group and were given TI-92s for the duration of the unit. The other class became the control group and had access to Texas Instruments TI-82s. Each class was taught a unit on linear functions.

Comparisons were drawn between the CAS group and the non-CAS group in relation to the three task foci. In addition, because little is known at present about the use of a CAS at the high school level, general issues concerning its use in the teaching and learning of mathematics at this level were explored and documented.

The learning outcomes were negotiated and developed by the two teachers and the researchers to ensure that outcomes included some focus on the three key aspects identified above: symbol manipulation, formulation, and problem solving. Pre and post tests for the unit based on the agreed learning outcomes were developed. Results from pre-tests were used to determine the comparability of classes and act as a covariate for analysis of treatment effects.

The post tests without use of the CAS were used to measure the immediate effects of CAS use on student understandings and ability to perform paper and pencil tasks related to symbol manipulation techniques, equation formulation and problem solving. A delayed post test is planned for the second term of 1998. Those classes taught with CAS were given an additional CAS 'active' test which permits the use of CAS. Results from the tests provided information on student ability to use the technology, on student ability to adapt to the new notational system required for operation of the CAS, and on general problems and difficulties experienced. Teachers and students were asked to keep a journal of classroom observations and their reflections on these observations and general issues of CAS classroom use. In addition, the research team made their own classroom observations of classes.

Results and Discussion

Task types and CAS use

In skill mastery tasks throughout the instructional unit, the CAS symbol manipulator provided an additional means of student 'checking' their answers. Many students commented that they found it very useful to have an electronic 'answer page', and there was evidence from student journal comments to suggest that this provided increased motivation for completing by-hand manipulation exercises within the unit.

Post-test results suggest that students in the CAS group were more inclined to persist with algebraic methods (usually by performing the same operation on both sides of an equation) for equation solving tasks, particularly when the level of difficulty of the manipulations increased. This was true even for test items where CAS use was not available (Table 1). It was also noted that the 'backtracking' method was more prevalent in the non-CAS group post-test results.

Sample Problem 1 - Solve for m: $\frac{3m+2}{5} = -4m$

Method	CAS Group	Non-CAS Group
Algebraic (successful)	*7*	*0*
Algebraic (unsuccessful)	*12*	*6*
Backtracking (successful)	*0*	*0*
Backtracking (unsuccessful)	*0*	*5*
No Attempt	*6*	*12*
	25	*23*

Table 1 - Student Response Summary for Sample Problem 1

In the post-test tasks requiring formulation (where no CAS was permitted), the CAS group were also more likely to use algebra than arithmetic, that is, attempting to construct and then solve an equation rather than by using backtracking or 'Trial and Error' (Table 2). However, when their results were analysed further, students from the CAS group were no more successful in completing such tasks than students from the non-CAS group.

Sample Problem 2 - I think of a number, multiply it by 5, subtract 3 and divide by 2. The result is 4 more than twice than the number I first thought of. What was the number?

Method	CAS Group	Non-CAS Group
Algebra	*14*	*6*
Backtracking	*1*	*4*
Trial and Error	*4*	*5*
No attempt	*4*	*8*
	25	*23*

Table 2 - Student Response Summary for Sample Problem 2

There was a clear trend in the types of errors students from both groups made in equation solving tasks, with mistakes commonly due to poor order of operations skills, and an inability to deal with negative number operations. None of these mistakes arose when students chose to use the CAS to perform such tasks.

In CAS 'active' post-test questions requiring students to describe each step performed in an equation solving task (CAS group only), students who chose to use the CAS were spectacularly successful, while those who chose not to use the tool often did not complete this task well (Table 3). It appeared that the CAS symbol manipulator was effective in focussing attention on the effects of manipulating symbols in an equation, and in formalising the approach used.

Sample Problem 3 - Use your TI-92 to solve the following equations, and fill in the table below to show what operations you have performed.

(a) Solve this equation for x: $\frac{4(x-7)}{3} = 3 - x$		
Operation	Performing operation	After Operation

Method	*Correct*	*Incorrect*	*TOTAL*
CAS used	*12*	*1*	*13*
CAS not used	*2*	*10*	*12*

Table 3 - Student Response Summary for Sample Problem 3

There were instances where students using the CAS-assisted approach would use an unconventional sequence of operations to solve the equations. In some cases this resulted in extra steps, but the students were still successful in completing the task. In contrast, students who did not use CAS were able to choose appropriate operations but were often unable to complete the manipulations. This result echoes findings by Coady and Pegg (1995) who, in attempting to identify the levels of higher order algebraic skills possessed by undergraduate mathematics students, noted that one major difference between student responses was in the 'ability to form a plan for the entire set of procedures required'. They noted that students often were able to begin procedures, but made manipulation errors or chose procedures that were unproductive.

Manipulation skills

There is a fear that frequent CAS use will reduce student's by-hand algebra skills.

> *'techniques and tedious calculations ... are as rigged and as tedious as finger exercises for the pianist, and just as indispensable'* Halmos 1990

In this study, the teachers expected that students in the CAS group would be able to perform the same by-hand manipulations as the students from the non-CAS group. The post-test results showed no significant differences between the groups (Table 4). This result mirrors the majority of studies where a CAS has been introduced.

	CAS Group	Non-CAS Group
Substitution/Expansion/Factorisation	8.0	7.2
Equation Formulation	1.2	0.8
Equation Solving Skills	6.7	6.8
Overall Post-test	15.6	15.2

Table 4 - Comparison of post-test mean scores (adjusted by the pretest covariate)

Rather than an over-reliance on the CAS, evidence from student journals, classroom observations, and teacher interviews suggest that under-use was more prevalent. Factors contributing to this include:

- some better students believed that the CAS was more tedious or cumbersome to use for the problems at hand.

  > *"I thought that the calculator was a waste of time because I already knew how to do everything by hand."*

- some students believed that the CAS was like 'cheating', and felt that it was not beneficial for them to rely on it. That is, they had short term access only, and the CAS in question was banned for Year 12 examination purposes.
- some students expressed frustration with the notation system or hardware/software.

  > *"I can do these in my head and get them right according to the book (text) but on the calculator I get them (some) wrong."*

- some students were less able and/or poorly motivated in mathematics classes, and saw the CAS as an additional burden. This was reflected in test results and student journals.

  > *"To [Teacher Name],*
  > *[Student Name] tried to work the super gadget, and got nowhere. I tried after, then her brother. None of us could make sense of it."*
  >
  > (signed by parent)

- teacher beliefs impacted how enthusiastically students took up calculator use. The calculator was only available to students during the study and therefore the teachers felt it was unwise for students to become dependent on it.

Notation issues

Mathematics has a complex set of notations in common use. The school curriculum has until recently carefully controlled the notations which students use at each stage of schooling. This changes with the use of CAS technology. A CAS uses symbols that they would not normally be exposed to in Year 8-10 mathematics, and in this unit students were introduced to notation that has a number of critical differences to the traditional notation. Some examples of syntax used by students in this unit are given in Figure 2.

solve(3m+2)/5=-4m,m)   *(3m+2)/5 | m=3*   *factor(2a+2b=7)*

Figure 2

Students were occasionally frustrated by the error messages which appeared. For example, a student entered **expand(b(b-2))** and the calculator returned the error message "Invalid implied multiply". Other students commented on how the calculator's auto-simplify feature would alter the form of the expressions or equations that they had entered

> *"After completing the sheet, I find it really annoying the way the calculator changes what we type in, into its own little order"*

Our experience with the Technology Enriched Algebra project (Tynan et al. 1995) and observations from this study suggest that students are more adaptable with notation than we had thought, and indeed by the end of the study period most students were aware of the meaning of error messages and had adapted to the notational demands of the calculator (at least as far as the content of this unit was concerned).

CAS and Curriculum Change

Some of the tasks in the unit were designed to make best use of the technology's 'what-if' features. These tasks support inquiry-based teaching approaches, rather than direct instructional methods. We found instances where the style of the tasks conflicted with the task type to which the students and teacher were familiar (see Artigue 1996 for further discussion of this issue). Tasks requiring open-ended student investigation were in some cases less successful than those tasks requiring a more limited but repetitive student response.

It may well be that the sort of curriculum change in which CAS becomes an indispensable tool will be a slow evolution rather than a revolution (Mann, Rothery & Sato 1997). Our experiences in this study lead us to believe there are a large number of factors that can affect the pedagogic impact of CAS at this level:

- the amount of student access to tool;
- teacher knowledge, beliefs and preferred style;
- student algebraic skills and preferred learning style;
- student motivation to succeed/persist;
- structural factors - competing pressures on student/teacher time;
- level of communication between research team and teachers;
- level of support for teachers;
- task type design - what interaction is encouraged between user and CAS;
- task type balance open-ended and more routine tasks.

Conclusion

CAS did appear to have some impact on student preferred methods of equation solving, and there was evidence that CAS helped students focus on choosing appropriate manipulations. Most students were enthusiastic about using CAS as a dynamic checking device, and post-test results indicate CAS use did not reduce student's by-hand algebra skills.

It is clear that the demands of the current courses, and the need to at least cover all of the existing curriculum material as well as more CAS-active tasks will continue to be a problem for teachers, and further work is needed to identify the tasks, strategies and balance of by-hand and CAS work appropriate at this level. The pedagogic impact of CAS at this level will be affected by many variables other than the presence of the technology itself, and attempts to control all these variables in any teaching experiment are at best problematic.

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Computers are taking mathematics into the next century: Gender differences in the attitudes of secondary mathematics students to the use of computers

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Secondary students who participated in a computer enhanced mathematics program expressed positive attitudes about the use of computers. They viewed computers as a source of pleasure, success, relevance and/or power in mathematics. Girls were more likely than boys to qualify their support for the use of computers and more likely to view computers as a source of success in mathematics. Boys were more likely to claim that computers brought pleasure or relevance to mathematics learning.

Background

Computers and other learning technologies continue to be advocated by mathematics educators and researchers as providing opportunities for new ways of learning and doing mathematics appropriate to the times (for example, Arnold, 1997). Few studies have explored gender differences in attitudes to the use of computers in mathematics. This paper reports on the results of a qualitative analysis of responses to the open ended questions in a questionnaire administered to year 8 and year 9 students who regularly used computers for mathematics. Gender differences are discussed.

Research over the past decade has consistently found that males use computers more than females (for example, Rocheleau, 1995). Studies of attitude to computers have in general found little evidence of gender difference, but the results of these studies have depended on definitions of attitude and the age of participants in the study (Kay, 1992).

Some studies have investigated changes in attitudes to mathematics due to the use of technology (Dunham, 1991), however little attention has been paid to gender differences and only a few studies have explored the reasons for these attitudes (Rowe, 1993; Schofield, 1995; Kissane et al, 1995). Dunham (1991) conducted an experimental study of tertiary pre-calculus students using graphing calculators. She found that males improved their confidence in mathematics relative to females and females improved their performance relative to males. Boers and Jones (1993) also found evidence of tertiary females performing better than males in a program involving graphing calculators. Boers and Jones attributed this finding to a difference in attitude such that, the boys relied too heavily on the calculators, whilst the girls, concerned about cheating, concentrated on pen and paper methods.

In an ethnographic study of primary aged children using laptops and LOGO across the curriculum, Rowe (1993) found that a higher proportion of the most effective users of computers in comparison to least effective users believed that the computer helped them in their learning, put them in control of their learning and was an asset for their future. She did not include a gender analysis of these beliefs but did reveal that there were more girls than boys who were rated as effective users. Schofield (1995) conducted an ethnographic study of secondary students engaged in trialing a computer aided geometry instruction program. She found that there was an increase in "friendly competition" particularly among the boys and more able students that contributed to students' positive and negative attitudes. She also found that the students enjoyed the lessons because there was an increased sense of control (over their own learning), an association with playing games, a sense of personal challenge, an increased opportunity to express negative feelings and a decreased fear of embarrassment. She did not provide a gender analysis of these factors but did argue that these factors were related to the nature of the software and the pedagogical style embedded in the software and the classroom.

Kissane et al (1995) conducted a study to determine whether tertiary students' views on the use of graphics calculators fitted with previously developed metaphors concerning the use of technology. These metaphors were: laboratory (using it to explore mathematical ideas and learn better), tool (using it to do mathematical tasks), teaching aid (using it to help with instruction), curriculum influence (a new element of content, teaching, learning and assessment), cheating device (a negative consequence of using technology), and a status symbol (a focus on the features of the technology). It was reported that the majority of responses reflected a positive view of the use of graphics calculators, however only 39 out of 73 students surveyed provided written comments and there was no gender analysis. The most common responses fitted with the metaphors laboratory, tool, curriculum influence and a new metaphor that was created, "nuisance".

Method

The sample: Students from two mathematics classes from a lower/middle class secondary school in Melbourne participated in the study. A total of 49 students (17 girls and 32 boys) from a year 8 class and a year 9 class completed the questionnaire. The ratio of boys to girls in the sample was almost 2:1. The bias was due to the huge majority of male students in the year 9 class (18 boys and 7 girls).

The year 9 class in the study was a "laptop class" with all students in this class leasing a laptop computer from the school. These students were in the second year of the laptop program and elected to be in the program at the beginning of the previous year. The year 8 class was a "high achievers class" with a program that included regular use of computers. Two mathematics lessons each week were timetabled in a computer laboratory. Some students in the year 8 class had purchased a laptop computer to use. The year 8 students were in the first year of the program. At the time that the questionnaire was administered, the year 9 students were working with a dynamic geometry application, *The Geometer's Sketchpad* and had been using *Excel* in the previous semester. The year 8 students had been creating a presentation of the solution process for multi-step equations using *Powerpoint* and were completing exercises in financial arithmetic using *Excel*.

Questionnaire and analysis: A questionnaire was designed to gather data about computer use, attitudes to computers and attitudes to computer enhanced mathematics. Three open ended questions were included in the questionnaire to gather information about students' attitudes to the use of computers in mathematics. These were:

- Do you think it is a good idea to use computers as part of maths? Explain your answer.
- What do you like most about using computers in maths?
- What don't you like about using computers in maths?

This paper report the results of analysis of responses to these questions. The responses were analysed qualitatively. The responses to the first question were coded numerically.

(no; yes; sometimes or 'yes and no'). Trends in gender differences are identified since a statistical comparisons of the frequency distribution was not valid due to the small number of responses in some cells.

The qualitative analysis involved a process of data reduction to analyse students' opinions about the use of computers in mathematics. A "bottom up" or inductive approach (Miles & Huberman, 1994) to coding the data was used. The process of coding for data reduction involved three stages: creating codes, pattern coding to create categories and identify dimensions within these categories and finally tracing relationships between categories and identifying emerging themes (Miles & Huberman, 1994). The responses to the questions were treated as one data set, rather than 3 separate sets of data. Students' opinions about what they liked the most and what they did not like about using computers in mathematics were often different sides of the same coin. For example, one student believed that computers were "faster, easier and more efficient" whereas another student wrote "I don't like using Geometers (sic) Sketchpad because ... it always stuffs up." As part of the process of data reduction these two responses were initially coded "quicker" and "doesn't work properly" respectively and later in the process were identified as examples of the positive and negative dimension of responses that were grouped together (pattern coding) and categorised *technology*.

The number and sex of students for each code and category (including the positive and negative dimension) were recorded. It was not appropriate to use statistical tests for analysing gender differences because the number of responses in some cells was too small to produce valid statistical comparisons. However, the major variations in the pattern of responses between the girls and the boys are described.

Results

Do you think it is a good idea to use computers as part of maths?

Most students (78%) thought that using computers in mathematics was a good idea, though a higher proportion of boys than girls believed this. See Table 1. A higher proportion of girls believed that it was only a good idea sometimes and there was a higher proportion of boys who did not think that it was a good idea.

Table 1: Do you think it is a good idea to use computers as part of maths?

Response	Female		Male		Total	
	N	%	N	%	N	%
no	1	5.9	4	12.5	5	10.2
yes	11	64.7	27	84.4	38	77.6
sometimes/yes & no	5	29.4	1	3.1	6	12.2
TOTAL	17	34.7	32	65.3	49	100.0

Analysis of open ended responses

The following discussion analyses the explanations given by students for the answer to whether using computers in mathematics is a good idea or not, and their likes and dislikes about using computers in mathematics. The process of pattern coding yielded 13 categories for the students' responses listed in Table 2. Each of these categories has positive and negative aspects. Table 2 shows the number of responses from girls and boys in each category.(Note that the totals in the column do not add to the number in the sample. This is because students gave responses to 3 questions and some responses contained more than one idea and so were coded in more than one way.)

Table 2: Categories for open ended questions and number of responses

CATEGORY	POSITIVE RESPONSES			NEGATIVE RESPONSES			TOTAL
	female	male	total	female	male	total	
Technology	5	10	15	7	8	15	30
Pleasure	6	18	24	1	5	6	30
Pedagogy	2	6	8	2	1	3	11
Learning Aid	7	2	9	2	0	2	11
Tool For Thinking	2	3	5	2	1	3	8
Tool For Doing	0	2	2	0	1	1	3
Computer As Teacher	0	1	1	1	1	2	3
Maths Difficulty	5	6	11	2	4	6	17
New Medium	7	6	13	1	0	1	14
Computer Skills	6	15	21	1	0	1	22
Maths Is Changing	1	1	2	0	2	2	4
Maths Content	0	4	4	2	1	3	7
Organis'n & Manage't	2	1	3	3	2	5	8

The largest numbers of responses were categorised as *technology* or *pleasure*. For the category of *technology* students supported their opinion about the use of computers in mathematics, their likes and dislikes by referring to the attributes or quality of the *technology* and its effect on productivity, such as the speed with which they could do things or the weaknesses of computers. For example positive comments included statements such as "using Excel and powerpoint because it takes shortcuts for you" (year 8 girl) and negative responses included statements such as "They take too long to start up" (year 8 girl). Students, especially boys, commented that they liked or disliked using computers in mathematics according to the amount of *pleasure* that they derived. This was expressed in terms of the degree of challenge offered by computers, the level of interest or the amount of enjoyment or fun. For example, "I think its good because the computer makes the subject more enjoyable thus causing a greater will to learn" (year 9 boy) and "I don't mind using sketchpad because it is an interesting program to use" (year 9 boy). *Pleasure* had the highest number of positive responses.

The category with the next highest number of responses and the second highest number of positive responses was *computer skills*. That is, the students also believed that the use of computers in mathematics gave them the opportunity to learn *computer skills*. These views were often related to their perception of the relevance of computers for their lives, for example, "Yes. Because you can use excell (sic) and today, computers are taking the world into the next century" (year 8 boy) and "Yes I do because at the same time we are learning how to use various software" (year 9 boy). One student did not see this benefit: "All we learn is how to use the program" (year 9 girl). Boys were over-represented in the category *computer skills*.

The categories with the smallest number of responses included *tools for doing*, *computer as teacher* and *maths is changing*. This is perhaps surprising given the emphasis of mathematics educators on tools and changing curriculum practise. Examples of responses for *maths is changing* included: "In todays (sic) society many school subjects are changing with the times.... I'm glad that now we use computers for everything and maths isn't left out" (year 9 girl). One student did not believe that maths had changed enough: "No because the teacher doesn't know how to make good use of it" (year 9 boy).

A larger number of responses were categorised as *maths difficulty*. Students commented on mathematics being easier because of the use of computers and thus saw its benefit in these terms. Some students thought that particular software introduced new difficulties and others saw that computers were not always the easiest way. For some students the benefits of using computers in mathematics was more to do with a new way of producing text (typing instead of writing) and presenting work (*new medium*). But other students indicated that using computers did not always make producing text easier or better. Girls were over-represented in both these categories.

Relationships and emerging concepts: Relationships between the categories were formulated by interpreting the meaning of responses. For example responses categorised as a *tool for thinking* were interpreted as making the *maths less difficult*. Positive and negative responses from students that show this relationship included: "I like using the excel program because you put in the formula and the computer does the work for you" (year 8 boy) and "Computers help maths with the things like calculations in Excell and sketchpad, but I don't think that they are vitally important as these calculations can be done in your head.... We depend too much on them giving us the answers" (year 9 girl). The relationships between categories and four emerging themes are shown in a concept map (Figure 1).

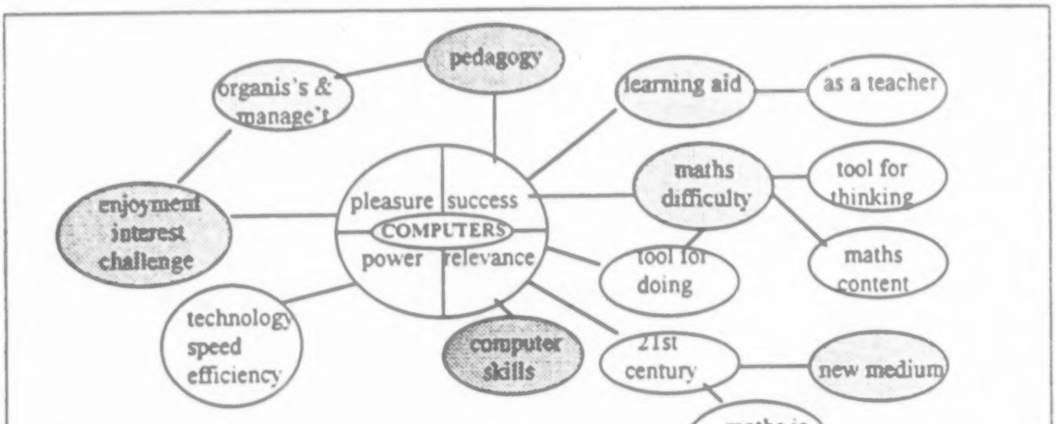
pedagogy
organis's & manage't
learning aid
as a teacher
maths difficulty
tool for thinking
enjoyment interest challenge
pleasure
success
COMPUTERS
power
relevance
tool for doing
maths content
technology speed efficiency
computer skills
21st century
new medium
maths is changing
☐ girls over-represented
▨ boys over-represented

Figure 1: Attitudes to the use of computers in mathematics

The other categories that could be associated with *maths difficulty* included *tool for doing*, *maths content*, *learning aid* (for example, "Using computers in maths ... makes

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it easier for me to understand what I'm doing." - year 9 girl), *computer as a teacher* and *pedagogy - new ways of learning* (for example, "Doing projects cause then you can use most of the software packages eg. Paint, word, exel." - year 9 boy). These relationships show that using computers in mathematics were a source of success or enhanced performance in mathematics. Conversely negative views in these categories were seen as contributing to a belief that computers were a source of failure or at least not enhance performance in mathematics.

Another set of categories indicated that students viewed computers as a source of pleasure or otherwise in mathematics. These categories not only included the one concerning *pleasure* as enjoyment, interest and challenge, but also *pedagogy - new ways of learning* and *learning organisation and management*. The category concerning *pedagogy* related to pleasure because some of the responses in this category made explicit or implicit causal relationships to enjoyment: "I like using the laptops because it varies from pen and paper which gets boring sometimes" (year 9 boy). Similarly responses in the category *learning organisation and management* implied that they are happy or not about the way their learning was organised when using computers: "We don't get to do what we want. We always have to do the work" (year 9 boy).

The idea of the future and the 21st century brought together another set of categories. Some students who made responses categorised as *computer skills* made explicit reference using these skills in the future, or for other subjects. More than one response implied that the usefulness of the software was important. Similarly the responses categorised as *tool for doing* implied usefulness and relevance. The idea of the future was also present in responses categorised as *maths is changing*. Responses concerned with using computers as the *new medium* implied an acceptance of changing technology and the improvements that can be made to presentation. The categories *computer skills, tool for doing, new medium* and *maths is changing* together indicated that students believed that using computers in mathematics was a source of relevance for mathematics.

The final category, *technology* (speed and superiority) sat on its own. The responses in this category indicated that students believed that computers were just better (or not). For some students this implied an over-riding concept of computers as a source of power. For example one student's response to the question about what he liked wrote: "POWERPOINT and MICROWORLDS. They rule" (year 8 boy).

To summarise the main themes then, students who believed that using computers in mathematics was a good idea viewed computers as a source of success in mathematics, a source of pleasure, a source of relevance, a source of power or a combination of these factors. Those students who did not think they were a good idea or wished to qualify their approval indicated that using computers in mathematics does not necessarily lead to success in mathematics, provide pleasure or relevance or yield power.

Gender differences: The differences in proportional representations of boys' and girls' responses were mentioned above. It needs to be remembered that this is a small sample in which the boys out number the girls almost 2:1 and some categories have very small numbers of responses. Table 2 provides information on the number of responses from boys and girls in each category. Girls and boys were represented in all categories except *tool for doing* for which only responses from boys were recorded. However there was a very small number of responses in this category. There were seven categories for which there was a disproportionate representation of boys or girls. Those categories in which females or males were over-represented are highlighted in Figure 1.

Girls gave disproportionately more views about computers:

- as *technology*, especially the negative view, that is that computers were not always the best or the fastest and that they were not infallible;
- helping with learning and understanding (*learning aid*), both positive and negative;
- making mathematics easier (*maths difficulty*);
- improving the presentation of their work, (*new medium*) - interestingly it was a girl who contributed the only negatively coded response in this category; and
- the way that learning was organised and managed in the classroom.

In summary, girls were more likely to give responses that indicate a belief that using computers in mathematics may enable them to succeed or enhance their performance in mathematics and they are more likely to value computers as a new medium and hence a source of relevance for their future. Some girls also see that using computers in mathematics does not necessarily provide them with power.

Boys were over-represented in the *pleasure, pedagogy* and *computer skills* categories. They provided disproportionately more comments than girls that computers:

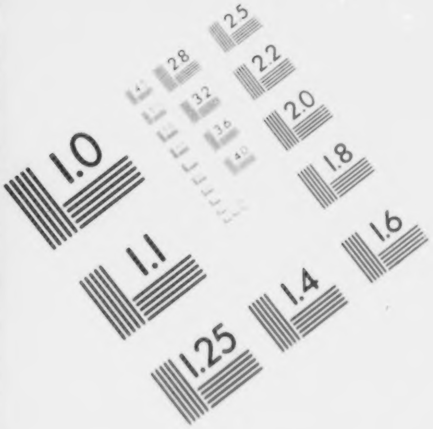
- were enjoyable, interesting and challenging (and negative comments in this category);
- brought positive changes to teaching and learning activities in mathematics; and
- enabled them to learn to use software that was useful for their future or other subjects.

Figure 1 shows that boys viewed using computers in mathematics as a source of pleasure and a way of making mathematics relevant. Computers as a source of success in mathematics or enhancement to their performance was a more indirect relationship concerning changing the ways of learning and it seems just as likely that changing the ways of learning may be more directly related to pleasure for this group of boys.

Discussion and conclusion

The results suggests that whilst both girls and boys would like to use computers as part of their schooling, they have different reasons and perhaps experiences of using computers in mathematics. There were some consistencies between the categories and themes that were identified in the current study with previous studies (Rowe, 1993; Schofield, 1995; Kissane et al., 1995). However some factors from previous studies were not evident in students' responses, for example, the ideas of control over learning and no fear of embarrassment (Schofield, 1995). The emphasis on computer skills made by boys was not identified in previous studies. The teaching and learning approach used by the teachers or the type of software that were used may explain the difference in findings. A comparative study of the introduction of computers (Goodson et al, 1995) found that particular implementations of computers are related to subject culture, defined as teaching practice.

Gender similarities and differences in student explanation of their views about using computers in mathematics were identified. Boys were more likely to be positive and negative about the use of computers in mathematics, and girls more likely to qualify their support for the use of computers in mathematics. The girls were more likely to associate the use of computers in mathematics with success, whereas boys were more likely to associate computers with pleasure and relevance of developing computer skills. These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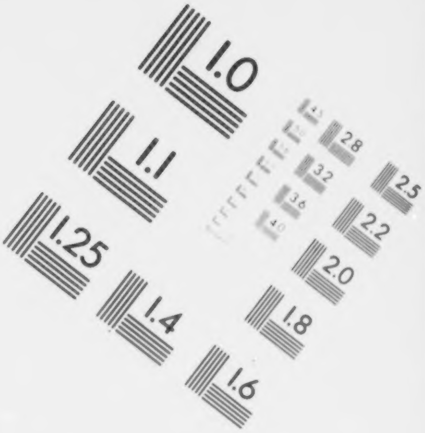
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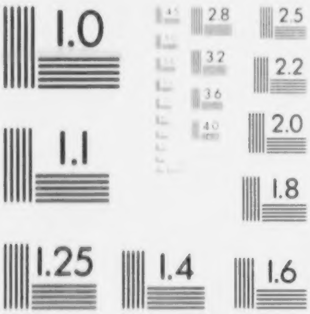
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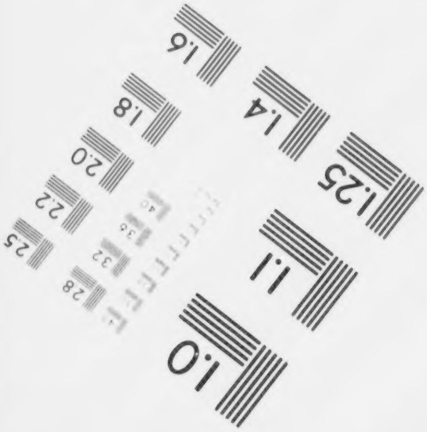
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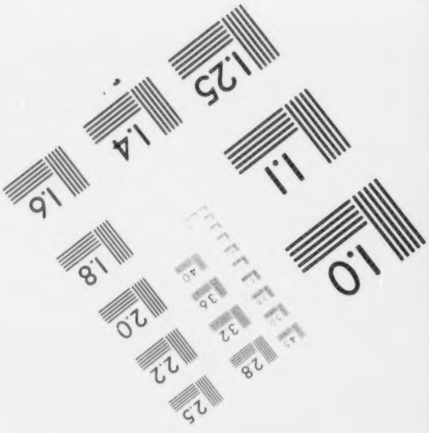
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results indicate that an implementation of computers in mathematics programs that emphasise the learning of particular computers skills and knowledge of a range of computer software is likely to be favourably received by boys. Whilst some boys did acknowledge that using computers enabled them to succeed in mathematics, a use of computers in mathematics that includes activities that boys find enjoyable is likely to enhance their attitude to mathematics. Alternately an implementation of computers in mathematics that focuses on improving understanding of mathematics or eliminates difficult tasks is likely to be viewed favourably by girls who will see that the use of computers in mathematics provides them with an avenue to success, though some girls will remain cautious about relying too heavily on computers.

These findings, and the extent to which they differ from other studies, indicate that classroom culture, the nature of the software, the curriculum context and the teaching and learning activities may play a role in determining the way that students view the use of computers in mathematics. A study that includes classroom observation and embraces these factors to provide some explanation for the differences in opinions expressed by boys and girls is the next stage to research.

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The Mathematics, Technology and Science Interface: Implementation in the Middle School

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The concepts of *middle-schooling* and *curriculum integration* were examined in this study, the aim of which was to provide a research base to inform the development and implementation of models of learning in mathematics, technology and science. A number of the case studies of integration conducted with students aged 11 to 15 years are reported, and issues arising from integrated practices are discussed along with their implications for teaching and learning.

Background

In Australia, as in many other parts of the world, the transition of students from primary to secondary school takes place around the age of 12 years. This transition involves a host of changes for students – from the single-teacher, single classroom, continuous curriculum environment of the primary school to that of the secondary school, where the discipline-based, many-teachers system involving discrete lessons taught in a number of classrooms or laboratories prevails. Recent research (Cormack, 1996; Hargreaves, Earl & Ryan, 1996; Speering and Rennie, 1996) has revealed the impact such student lifestyle changes have on student-teacher relationships – changes which have the potential to make these middle years of schooling potentially stressful and alienating for those students.

One of the recommended ways of addressing this problem is to provide a curriculum that is relevant, negotiated and integrated (Cumming, 1996). How to achieve this goal is currently the source of considerable debate among educators as are many other issues related to curriculum integration – for example, whether the curriculum should be primarily discipline-based, issues-based or project-based (Cumming, 1993). Bean's view is that it is incumbent on educators to provide early adolescents with an integrated approach to teaching and learning – that is, one which provides a curriculum that motivates students, where real life questions are posed, and where wholeness and unity are promoted rather than separation and fragmentation (Bean, 1991, p.9).

Why Integrate?

The potential for integrating science, mathematics and technology is well recognised (LaPorte & Sanders, 1993; Wicklein & Schell, 1997). One study (LaPorte & Sanders, 1993) involved middle school teachers integrating these subjects through technology-based projects. The authors concluded that activities depending on students applying science and mathematics knowledge to solve problems in technology offer a practical, meaningful and motivating learning context. Also, a number of case studies of approaches to integrating mathematics, science and technology education (Wicklein & Schell, 1997) found that three primary factors significantly affected the success or failure of the curriculum including "(1) teacher and administration commitment to the integration approach, (2) innovation and effort in curriculum re-design, [and] (3) administration and teachers' co-ordination of the integration plan" (p. 6).

Several authors have classified integrated teaching practice into various models, stages and continua, with a variety of associated terminology. Fogarty (1991), for example, proposed ten models of integration, ranging from the traditional fragmented model to those that connect, sequence, nest, share, web, thread, integrate, network or immerse the learner in curricular integration. Reflecting on the experience of a team of teachers who worked with her on an integrated project, Drake (1991) describes a progression in the process of curriculum development through multidisciplinary, interdisciplinary and transdisciplinary approaches.

Marsh (1993) suggests that the various forms of curriculum integration can be considered as a continuum beginning with "discipline-based options" with separate subjects taught at different times. The first step towards integration is "parallel-disciplines" where content is sequenced to correspond with related content in other disciplines. Marsh then describes "multi-disciplinary thematic approaches" where the various subjects contribute to a central theme and "interdisciplinary concepts and topics" where discipline concepts are chosen because of their direct relevance to the theme. "Internal orientation" encompasses activities which are jointly planned and implemented by students and teachers and the final variation of curriculum integration on the continuum is "whole school integration" which involves a total transformation of the learning environment as practised in Waldorf schools (Barnes, 1991).

Panaritis (1995) advises that teachers should be careful not to be driven by models such as those proposed above because the underlying distinctions are often "either fundamentally irrelevant or hopelessly arcane" (p. 627). Panaritis suggests that it is demoralising for teachers to have their efforts classified as "merely parallel" or "not *truly* interdisciplinary" (p. 627). Another criticism comes from Case (1994), who warns that policies advocating curriculum integration are little more than slogans. The kinds of programs which result can be ill-conceived and disconnected from real academic subjects (Brophy & Alleman, 1991).

There is conflicting evidence about the success of integrated programs. According to Vars (1991), since the 1940s more than 80 normative or comparative studies have reported that students in various forms of integrated programs performed better, or as well, on standardised achievement tests than students enrolled in separate subjects. Marsh (1993) tracked some of the major research studies on integration from the USA, the UK and Asia over the past half century and concluded that although the earlier studies "gave the impression that curriculum integration had many positive elements over single discipline teaching, there is a dearth of evidence of a positive or negative nature over recent years" (p. ii). While some empirical studies have shown that students can learn mathematical and scientific concepts in an integrated manner (Richie & Hampson, 1996; Roth, 1993), there also is evidence that some students have difficulty in grasping the content of integrated courses (Richie & Hampson, 1996; Wicklein & Schell, 1997).

Internationally, the past decade has seen several influential documents advocating integration of science, mathematics and other content areas. In the USA, *Science for All Americans* (Rutherford & Ahlgren, 1990), *Everybody Counts* (National Research Council, 1989), *Curriculum and Evaluation Standards for School Mathematics* (National Council for Teachers of Mathematics, 1989), and *The National Science Education Standards* (National Research Council, 1996) all stress the interrelatedness of mathematics and science and the implications for curriculum and instruction (Lonning & DeFranco, 1994). In the United Kingdom, the National Curriculum Council distinguishes cross curricular elements in terms of dimensions, skills and themes (Nixon, 1991). Similar trends are evident in Australia, for example, in the recently published draft of the *Curriculum Framework for Western Australia* (Curriculum Council of Western Australia, 1997). The Western Australian framework describes mathematics, science and technology as three of eight separate learning areas and advocates an integrated approach to curriculum delivery:

> While eight learning areas have been identified, knowledge, skills and values should be integrated across all learning areas. Students should be given frequent opportunities to see the connections between different areas of knowledge and endeavour. They should be encouraged to understand the arbitrariness of any division of knowledge into learning areas and subjects or other categories; to appreciate the interconnectedness of all knowledge; and the indissoluble relationship between knowledge and values. (Curriculum Council of Western Australia, p. 27)

While the new Western Australian framework exhorts teachers to teach in a more integrated way, the culture of the middle years of schooling in this State remains predominantly discipline-based in high schools and to some extent in the upper years of elementary or primary schools. There are, however, some interesting exceptions—schools which have managed to incorporate aspects of integration into their programs. It is these exceptions which form the focus of this study of integrated teaching of science, mathematics and technology in Western Australian classrooms in Grades 7-9.

The Study

A number of Western Australian schools which have successfully introduced integrated approaches to curriculum were the subject of this study. This paper examines such innovation and describes the different ways that the integration of mathematics, science and technology has been achieved. In addition, the paper examines five issues raised by teachers during interviews. These include the process of getting started, implications of integrated practice for teachers and students, structural implications, and the breakdown of departmental structures.

Method

The data were collected from 16 schools, including government and independent high schools and primary schools in the metropolitan area and country districts of Western Australia. The schools were selected on the basis of the existence of some integration between mathematics, science and technology in the middle school years, or that teachers were interested in integration. Four of the 16 schools were involved in the research as in-depth case study schools.

Each of the 16 schools was visited by one of the researchers who interviewed teachers about the integration at their school. The researcher questioned the teachers about integration that had occurred or had been attempted at the school, the teachers' opinions about the factors that enabled or inhibited the integration, and the perceived advantages and disadvantages in the process. The researcher also observed mathematics, science or technology lessons. Data were collected by field notes, and a summary of the school visit was compiled. The teachers were sent a copy of the summary and invited to comment and provide feedback.

The four case study schools were involved in more in-depth data collection, including interviews with from three to seven teachers and/or the principal, the collection of related documentation such as school policy documents, teachers' programs and inspection of student work, and the observation of between five and seven lessons. All interviews in the case study schools were tape recorded and transcribed and notes were taken during classroom observations. Each of the four schools was given a copy of the case study account with an invitation to provide feedback. The data from all of the 16 schools, including the four case studies, were analysed to provide the findings for this paper.

Forms of Integration

Different forms of integration were utilised by the 16 schools in this study, each form described briefly below.

Thematic Approach

A thematic approach to integrating mathematics, science and technology and other subjects was observed in two schools. In both schools a theme was used to integrate the curriculum, and the teachers responsible for each of the learning areas developed a program of work complementary to the theme. Examples of the themes used in these two schools included: decision making, the Olympics, relationships, systems, communication, popular culture, co-operation, Earth and people, and resources. Teachers commented on the different levels to which they were able to integrate their learning area topic with the theme, with most teachers expressing the view that they were able to make a number of significant links.

Cross Curricular Approaches

Several schools focused on cross curricular issues such as numeracy, literacy and computing skills, and teachers commented that these issues served as a basis to their integration between learning areas. For example, a technology teacher pointed out that literacy was clearly integrated into the technology course through the interpretation of the design brief and the written assignments, and that mathematics skills were utilised especially with measurement. Computing was also integrated, with students using word processors for the research assignments and programs such as CAD for the design of products.

Technology-Based Projects

One school used a technology-based project system of integration. Students were given a project to work on for 10 to 12 weeks which included a technological, a scientific and a mathematical research component. An example of a technology project brief was to "design and produce an electric powered vehicle that can climb a steeper gradient on the standard test track than anyone else's." The technology research component investigated traction options, materials and construction techniques, motor mounting options and power transmission systems. The mathematics research component investigated the effects of changing variables on standard Lego model hill climbers and recording, presenting and analysing their group's results from the time trials.

Competitions

Teachers at some of the schools mentioned that competitions such as the Solar Car Challenge and Science Talent Search provided good opportunities for integration between science, mathematics and technology. The Solar Car Challenge involved students in making a solar powered car and then racing the car against other students' creations. Students engaged in science through the investigation of solar energy, in technology through the construction and improvement of the vehicle, and in mathematics by working out the most efficient angle for the solar panel and the statistical analysis of racing trials.

Topic Integration

A mathematics teacher and a social science teacher in one school initiated an integrated project completely amalgamating the mathematics topic of Statistics and the social science topic on World Environment. The project ran for three weeks and required groups of students to (i) research a developed and (ii) a developing country and perform various statistical analyses on the data they collected (such as mean, mode, median and standard deviation) and present the results in a graphical form.

Integrated Assignments

Three schools used integrated assignments between learning areas that took between one lesson and two weeks to complete. For example, in one school, the mathematics teacher organised integrated mathematics and science investigation style assignments every five to six weeks. One of these investigations was on pendulums and involved the Year 8 students in developing a hypothesis and experimental procedure, statistically evaluating the data and presenting the results in graphical form.

Synchronised Content and Processes

Many teachers explained that the content in their different learning areas was synchronised as much as possible to allow similar content and processes to be taught at similar times. This involved the teachers in each school writing down everything they hoped to achieve in the year, making a map of the possible links and then rearranging the program where possible. One example of these links involved chi square analysis in mathematics/statistics and Punnett square problem solving in genetics. The teachers were able to synchronise the teaching of these concepts and explicitly link the ideas together.

Local Community Projects

Two schools had been involved in local community projects which integrated skills and content from a number of learning areas. One of these projects involved students, in liaison with a local shopping complex, providing plans for an improved arrangement of parking and pathways prior to major earthworks taking place. A second school said that they had worked with the local council to prepare an inventory of services provided to the community.

Common Teaching Approaches

One form of integration involved using the same framework for designing and writing up investigations in mathematics and science. Teachers said that using the same approach helped students to perceive similarities in the processes involved in working scientifically or working mathematically. One teacher commented that the staff members in her school tried to utilise student-centred teaching strategies as much as possible, and that this common base to their teaching approach provided a catalyst to integration of content and skills across the learning areas.

Natural/Informal Integration

Teachers in four primary schools found that much integration happened naturally because they taught the majority of learning areas to the same students, and "links just cropped up". In one school, technology was promoted as a vital learning tool to extend other subjects. For example, mathematics investigations were used in technology projects and the students explored such things as the cultural value and profit margin of the products. One group of students made a bird feeder by designing a template and then constructed the final product. Students' mathematics skills were assessed through this technology-based project.

Issues Arising from Integrated Practice

Interviews with teachers raised several issues about integration, including the process of getting started, implications for teaching, structural implications, implications for students and the breakdown of departmental structures. Each of these issues is discussed below.

Getting Started

Teachers mentioned several factors that helped initiate the process of integration, including prior planning involving all participating teachers. This created a collective vision and promoted collegiality. The importance of administrative support from people such as the principal was an important factor for motivating change. A caring ethos was regarded by some teachers as a positive factor facilitating the process of integration by encouraging better relationships between students and teachers. Professional development was also highly regarded by teachers who had the opportunity to visit schools where integrated approaches had been implemented.

Implications for Teachers

Teachers felt that commitment and passion were necessary for the challenging and innovative task of integrating across learning areas. The willingness to take risks was seen as a desired quality because of the need for teachers to forego their traditional practices, often established over long teaching careers. The ability to work effectively in teams was considered to be important for integration, especially in high school situations, because of the necessity to work closely with teachers from other learning areas. being recruited into a school knowing that integration was something that would be encountered was said by those in that situation to be an advantage because they 'knew what they were in for'.

Teaching experience in integrated situations and an intimate knowledge of the Westaer Australian *Outcome Statements* were also recognised as factors that facilitated integration. One principal noted that successful integration required teachers to manage a complex and delicate balance between knowing their own learning area deeply and knowing the specifics and commonalities between different learning areas.

Teachers felt they benefited by working in integrated environments because they could choose content to motivate and interest their students; it was a more economical way of achieving the outcomes; it helped them to co-operate and develop professionally, and they enjoyed working with teachers from other learning areas. Some teachers expressed frustration at the slow process of implementing integrated practices, the trial and error nature of the task and the increased work-load on top of their normal teaching duties. These factors could be stressful, especially for those in their first year of teaching.

Structural Implications

Ongoing collaborative planning time was probably the most frequently mentioned factor necessary for integration. A flexible timetable allowed teachers to take students on excursions, to team teach, and swap, extend or shorten lessons when necessary to allow for more integrated practices. Sharing materials with other schools, departments or learning communities and designating responsibilities to individuals in teaching teams was said to help reduce the heavy workload. Principals from two schools with a high number of new teachers each year stressed the necessity of providing induction sessions to orient new staff to integrated curriculum approaches.

Implications for Students

Teachers noted the advantages of integration for their students. The contextualised nature of problem solving was a highly-regarded benefit of integration, especially by mathematics teachers. Some noticed their students had better understanding of mathematics and science concepts when they had applied their knowledge to a practical task in technology and, conversely, the technology products were of a better quality when the students were able to use mathematics and science skills and knowledge to improve their designs. Teachers as well students appreciated the worldly application of the concepts.

To some teachers, an integrated environment meant that students perceived the common threads between the separate learning areas, and could better transfer knowledge and skills between subjects. Teachers were able to reinforce similar ideas in different teaching contexts and could also come to a consensus on how to teach common concepts. Other benefits included the enhancement of group working skills and creativity; increased co-operation and collaborative learning; successful behaviour modification among students; responsibility for their own work, and an increase in the mentoring between students of different ages. Difficulties for students included the lack of ability to plan and manage their time effectively for integrated projects, a lack of enjoyment of the work, and a realisation that other students didn't work hard enough to accomplish set tasks.

Breakdown of Departmental Structure

Frequently mentioned obstacles and difficulties to integrated teaching practice related to the breakdown of traditional departmental structures. Team teachers missed the specialist professional support of the traditional subject departmental groupings. Teaching in integrated situations often meant that teachers were working outside their subject speciality. These teachers said they lacked confidence in their ability to teach in this situation and, most importantly, they raised concerns about the implications for student learning.

The teachers in one primary school were concerned about integration because they valued science, mathematics and technology as different subjects with different orientations including processes, concepts, conventions and attitudes. These teachers believed that teaching in an integrated way might lead to science, mathematics and technology becoming an "amorphous mass". They explained that teachers need to be explicit and clear about what they are teaching and what outcomes are to be achieved.

Secondary teachers, especially those with a mathematics background, expressed concern at the heterogeneous nature of integrated classes and the difficulty of changing from traditional arrangements into homogeneous ability-level groups. Other teachers lamented that the integration they were involved in seemed contrived and imposed, and therefore the links between the subject areas were superficial and unrealistic.

The breakdown of departmental structure did not always have community endorsement either. Parents, teachers and the general community are secure in traditional structures and some teachers commented on a general fear of change.

Discussion and Conclusions

All forms of integration observed in the 16 Western Australian schools in this study maintained the discrete subject disciplines and integrated by means of themes, projects, assignments, synchronised content, cross curricular issues, competitions, school specialities, and teaching approaches. Synchronised content, or "parallel discipline design" (Marsh, 1993, p. 4), together with multidisciplinary thematic approaches which involve various subjects making contributions to a central theme, are small steps from traditional discipline based approaches (Marsh, 1993). The identity of each subject was maintained and subject teachers devoted varying amounts of time directly related to the theme. Marsh (1993) is of the view that it is very difficult to get teachers to undertake themes based on interdisciplinary concepts and topics where the disciplines become secondary to the theme. Many would argue that the integration of concepts in this way leads to more meaningful contexts for students and therefore enhanced learning possibilities (Bean, 1991). It is noteworthy that none of the schools in this study was involved in more consummate forms of integration such as interdisciplinary themes, student planned and implemented themes or whole school curriculum integration.

The results of this study highlight the nature of integration as a double-edged sword. On one hand teachers observed many benefits of integration for their students, including better understanding of mathematics and science concepts when applied to contextualised technology tasks. On the other hand, teachers raised concerns about departmental structure, their own insecurities, and the depth of content in each of the respective disciplines. Some teachers felt insecure when removed from their subject departments because they felt the robustness of the content was lost. It is pertinent that some of the primary teachers interviewed in this study raised concerns about mathematics, science and technology becoming an "amorphous mass" through integration. For many teachers, it threatened existing structures of power and control

Curriculum integration is not an end in itself, but a means of achieving basic educational goals. A fundamental goal of education in the middle school years is to deliver a curriculum which is relevant to students entering early adolescence. This is a difficult task for even the most motivated of teachers. This study identified several ways that teachers have been able to achieve this goal using several different forms of curriculum integration. While integration may not provide the ideal, nor the only way to solve the vexing problems associated with middle schooling, the evidence from this study suggests that some forms of realignment and reinvention of subject boundaries can provide enhanced opportunities for students experiencing this important stage of their schooling.

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Progressing through the van Hiele levels with Cabri-géomètre

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The influence of the dynamic geometry tool, Cabri-géomètre, on the learning of geometry was investigated. Twelve Year 7/8 pupils were divided into two groups on the basis of van Hiele pre-tests: Group I, pupils at Level 0 or 1, and Group II, the remaining pupils. Both groups completed six lessons with structured Cabri worksheets then Group II pupils completed several construction tasks, documented by means of Cabri files and tape-recorded conversations. All pupils increased their van Hiele levels for some or all of the concepts involved.

Cabri-géomètre™ is one of several dynamic geometry software tools which have created much interest amongst mathematics educators. Accurately measured constructions can be produced quickly, enabling explorations which previously would have been seen as too tedious and time-consuming. Use of these computer tools also overcomes the limitation of students' lack of precision which may result in invalid interpretations when working with pencil and paper. More importantly, continuous transformations of drawings are possible in the dynamic environment, allowing pupils to see relationships which otherwise may not have been apparent.

The van Hiele theory of geometry learning

Children's learning of geometry was investigated in the 1950s by the Dutch researchers, Pierre van Hiele (1957/1984) and his wife, Dina van Hiele-Geldof (1957/1984). Since then, their theory of levels of learning has been the subject of research in many countries. According to the theory, there is a hierarchy of levels through which children progress:

Level 1	Shapes are recognised by their visual appearance alone.
Level 2	One or more properties of a geometric shape are recognised.
Level 3	Relationships between properties are recognised.

Two further levels were identified by the van Hieles, but Level 4 would be reached by only a few pupils and Level 5 normally would not be relevant to secondary school geometry. Dina van-Hiele Geldof noted that it took 20 lessons to take her 12 year-old pupils from Level 1 to Level 2 and a further 50 lessons to take them from Level 2 to Level 3.

Aims of the study

The study was designed to investigate the following hypotheses:

1. Learning geometry with Cabri allows students to progress from one van Hiele level to the next more rapidly than suggested by Dina van Hiele-Geldof.
2. When using the dynamic environment of Cabri, students initially regress in the thinking displayed in their constructions to a level lower than their conventionally measured van Hiele level.
3. Learning with Cabri contributes to the understanding and use of correct geometric language.

The participants

The study was undertaken in a private girls' school in Melbourne with 12 members of an accelerated Year 7/8 class (ages 11-12). They had a wide variety of backgrounds at Grade 6 level and a wider than expected range of abilities for an accelerated class, although all were highly motivated. The girls had their own notebook computers, but had not used Cabri prior to this study.

Methodology

The study was essentially a case study. It was both quantitative, involving measurement of changes in van Hiele levels associated with the use of Cabri, and qualitative, documenting and analysing pupils' geometric language and methods of construction of geometric figures with Cabri. Data collection, in the form of students' test responses, Cabri files and taped conversations, took place over a period of four weeks. The methodology involved the following stages:

1. Geometric terms test
2. Van Hiele pre-tests
3. Cabri worksheet lessons
4. Cabri constructions for selected students
5. Van Hiele post-test
6. Further Cabri constructions for selected students

Geometric terms test: Pupils matched terms such as parallel, perpendicular, isosceles triangle, scalene triangle with appropriate drawings.

Van Hiele test: Written test developed by Lawrie (personal communication, 1997) from Mayberry's van Hiele test (1983), testing the concepts squares, right-angled triangles, parallel lines and isosceles triangles. On the basis of the van Hiele pre-test, the twelve pupils were divided into two groups:

Group I: Six pupils who were at Level 0 or 1 on all or most concepts.
Group II: Six pupils who were at Level 2 or 3 on all or most concepts.

The Cabri lessons (All lessons were 45 minutes and were spread over three weeks.)

Lesson 1: Cabri introduction. Students explored the menu options.

Lesson 2: Cabri exploration. Students were directed to investigate the difference in behaviour between basic point, point on object and intersection points. They also marked and measured angles, measured line segments, labelled points.

Lessons 3-8: Learning with Cabri. Students completed a set of structured activities which served the dual purpose of allowing them to become more confident in using Cabri while at the same time constructing or reinforcing their own knowledge of some basic properties of parallel lines, triangles and quadrilaterals. Teacher intervention while the students were using Cabri was minimal, the most frequent reason being to assist them when they had chosen an inappropriate Cabri point. Approximately 5 minutes were spent in class discussion at the end of each lesson reinforcing the properties the students had explored. These explorations were:

1. Angles in a straight line and vertically opposite angles
2. Angles in a pair of parallel lines cut by a transverse line
3. Angle sum of triangles and external angles of triangles
4. Angle sum of quadrilaterals and diagonals of quadrilaterals

Construction tasks: On completion of the Cabri worksheet activities, Group II students were required to construct the following drag-resistant figures in Cabri: right-angled triangle, rectangle, parallelogram, isosceles triangle. These tasks took one or two lessons. The pupils' construction steps were retraced in the saved files by means of Cabri's *History* option. Analysis of the constructions enabled the author to identify the van Hiele levels of thinking displayed in the Cabri constructions.

Van Hiele post-test: The Mayberry/Lawrie van Hiele test was administered as a post-test. (The students had not been told their scores on the pre-test.)

Further construction tasks: Following the Mayberry/Lawrie post-test, selected Group II students constructed letter A and 'House' shapes (Figure 1) in Cabri so that the shapes would retain their basic properties when dragged.

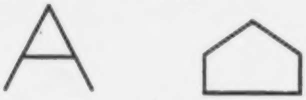


Figure 1. Letter A and House construction tasks

Results

Table 1 shows the pre-test and post-test levels for the four concepts for Group I students. All six students increased by one level in at least two of the four concepts and 15 of the total of 24 individual concept levels increased. All six students were now at Level 2 (Students C, D, I and K) or 3 (Students B and F) for Squares and only two students remained at Level 1 on each of the other three concepts. Student F had progressed from Level 0 to Level 2 for isosceles triangles. These increases, which occurred after six Cabri lessons (lessons 3-8), contrast with the reported 20 lessons taken by Dina van Hiele-Geldof to bring her 12 year-old pupils from Level 1 to Level 2.

Table 1. Group I pre-test and post-test van Hiele levels for four concepts

Student	Squares		Right-angled triangles		Isosceles triangles		Parallel lines	
	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test
B	2	3	1	1	1	1	1	2
C	2	2	1	2	1	2	1	2
D	1	2	1	2	0	1	1	1
F	2	3	1	1	0	2	1	1
I	1	2	1	2	2	2	1	2
K	2	2	2	2	1	2	1	2

Note: Shaded cells indicate where increases in van Hiele level have occurred

Table 2 shows a comparison of pre-test and post-test results for the four concepts for the Group II students. Five of the six students progressed from Level 2 to Level 3 in two of the four concepts, with Students G, J and L progressing to Level 3 on three concepts. In three cases, students satisfied the criteria for Level 3, but not for Level 2, for a particular concept and could not be classified. This may indicate that these students were in transition between levels. Given that Students A, E, G and H were already at Level 3 for one of the four concepts, increases occurred in 14 of the possible 20 individual concept levels. These increases, which occurred after seven or eight Cabri lessons, contrast with the 50 lessons taken by Dina van Hiele-Geldof to bring her 12 year-old pupils from Level 2 to Level 3.

Table 2. Group II pre-test and post-test van Hiele levels for four concepts

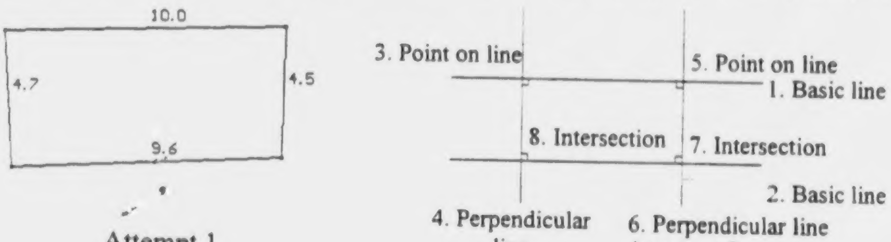
Student	Squares		Right-angled triangles		Isosceles triangles		Parallel lines	
	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test
A	2	3	NC	NC	2	3	3	3
E	2	3	3	3	NC	3	2	3
G	2	3	NC	3	3	3	2	3
H	2	2	2	3	2	2	3	3
J	2	2	2	3	2	3	2	3
L	2	3	1	3	2	2	2	3

Note: NC Not classified
Shaded cells indicate where increases in van Hiele level have occurred

The van Hiele levels displayed in the students' constructions

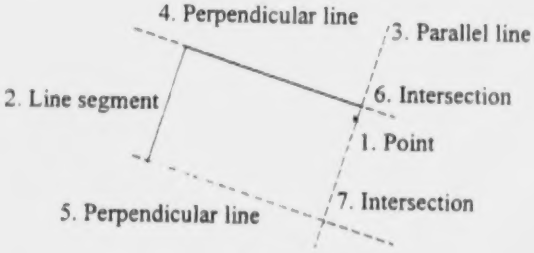
All Group II students, with the exception of Student E, used Level 1 by-eye strategies to varying degrees when constructing their first figure. When it came to other shapes, some of the students still attempted by-eye constructions before resorting to the Construction menu, even though they knew from constructing their first shape that by-eye constructions were "messed up" (Healy, Hoelzl, Hoyles and Noss, 1994) by dragging. Others, however, tended to be less dependent on by-eye methods and immediately explored geometric construction methods, confidently selecting the appropriate menu items. It was apparent also that the students' understanding of geometric language was developing through using the construction tools.

The following examples illustrate the progress in thinking which occurred. Student A's first attempt at the construction of a rectangle (Figure 2) was purely visual.


10.0
4.7
4.5
9.6
3. Point on line
5. Point on line
1. Basic line
8. Intersection
7. Intersection
2. Basic line
4. Perpendicular line
6. Perpendicular line

Attempt 1

Attempt 2


4. Perpendicular line
3. Parallel line
2. Line segment
6. Intersection
1. Point
7. Intersection
5. Perpendicular line

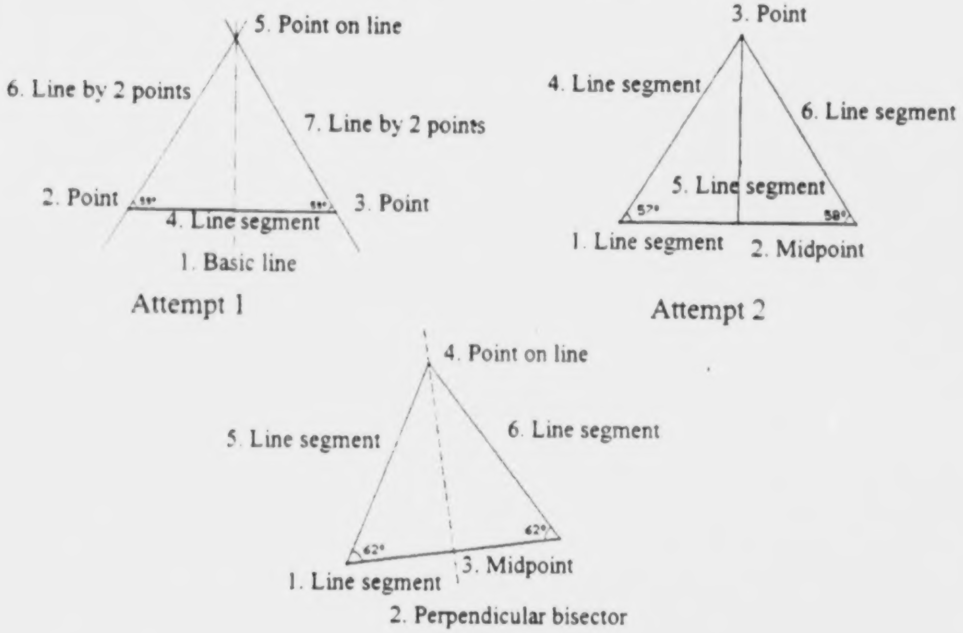
Attempt 3

Figure 2. Student A: Construction of a rectangle

Student A's second attempt was partly visual, with two *Basic lines* aligned parallel to the lower edge of the screen, but the vertical lines were constructed using the

Construction menu item, *Perpendicular line*. Her final, successful attempt was based on correct geometric construction using both *Parallel line* and *Perpendicular line* from the Construction menu. Student A no longer aligned her rectangle parallel to the screen edges.

A similar progression from purely visual to geometric construction is apparent in the construction of an isosceles triangle by Student G (Figure 3).


5. Point on line
6. Line by 2 points
7. Line by 2 points
2. Point
3. Point
4. Line segment
1. Basic line
3. Point
4. Line segment
6. Line segment
5. Line segment
57°
58°
1. Line segment
2. Midpoint
4. Point on line
5. Line segment
6. Line segment
62°
62°
1. Line segment
3. Midpoint
2. Perpendicular bisector

Attempt 1

Attempt 2

Attempt 3

Figure 3. Student G: Construction of an isosceles triangle

In her first attempt, Student G placed Points 2 and 3 by-eye, while in her second attempt, Point 3 was placed by-eye above the geometrically constructed midpoint of the base. In her final construction, which represents a transition to Level 3 thinking, she employed the Construction menu item, *Perpendicular bisector*, a term which she had not known on the Geometric terms pre-test.

The constructions of the letter A and House shapes showed a clear progression from purely visual Level 1 thinking to Level 3 thinking, where the students were giving careful consideration not only to the properties of the figures but to the relationships between the properties. The shapes were mentally dissected and reassembled, allowing the pupils to develop a logical ordering of the construction steps. The initial by-eye attempts provided the visual framework on which to build a geometric construction. Although the students occasionally lapsed back into by-eye methods during these constructions, in general they became extremely focused, confidently selecting the Cabri Creation and Construction tools and using appropriate geometric language. Figure 4 shows the first joint attempt by Students A and G for the House shape, which was partly visual but with geometrically constructed parallel lines.

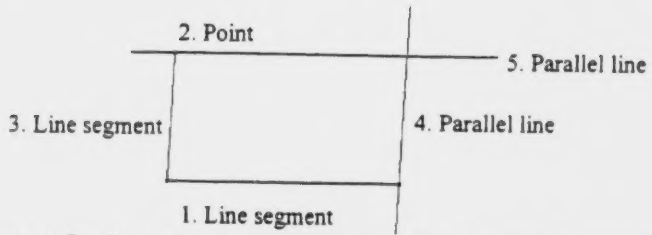

2. Point
5. Parallel line
3. Line segment
4. Parallel line
1. Line segment

Figure 4. Students A and G: Construction of House shape (Attempt 1)

Distortion of this construction when dragged enabled Students A and G to analyse the shape and think about the relationship between the properties of the component parts, resulting in the Level 3 construction shown in Figure 5.

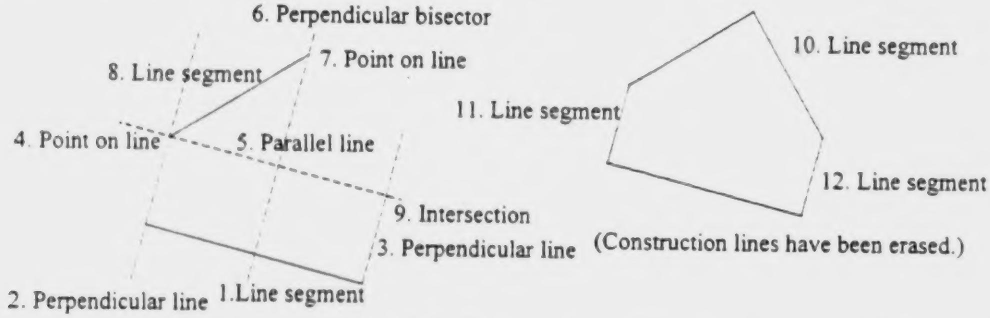

6. Perpendicular bisector
8. Line segment
7. Point on line
4. Point on line
5. Parallel line
9. Intersection
3. Perpendicular line
2. Perpendicular line
1.Line segment
10. Line segment
11. Line segment
12. Line segment
(Construction lines have been erased.)

Figure 5. Students A and G: Construction of House shape (Attempt 2)

Validation of constructions

The students all made at least some measurements, either of angles or sides to check their constructions, particularly where they employed by-eye methods. Usually these measurements were made before dragging the figures. As the students left their by-eye constructions behind and used the Construction tools, they generally no longer measured angles or sides and used the drag facility instead as a means of validating their constructions. They seemed to have realised that there was little point in making measurements if their figures were not based on geometric constructions.

Development of geometric language

The use of geometric language by Students A and G was evident throughout their construction of the House shape, and this played an important part in their ability to discuss and verbalise the next step in their construction. As new terms were learned they became incorporated into the students' language. "*Perpendicular thing*" said Student G, only to be corrected by Student A: *"A perpendicular bisector"*. The Cabri tools and the students' geometric understanding were becoming intertwined. This confident use of geometric terms supports the van Hieles' belief that knowledge and use of appropriate language is a prerequisite to the development of thinking skills and progression through the levels. Van Hiele-Geldof (1958/1984) notes that "concept and language can be distinguished, but cannot be separated. The thinking operation itself first has to be made conscious through language symbols and the language symbols are a consequence of the thinking operation" (p. 232).

Conclusions

The findings of this study support the hypothesis that using Cabri can indeed enable a more rapid progression through the van Hiele levels than found by van Hiele-Geldof (1957/1984). Pegg (1995, p. 93) notes that students must face a "crisis of thinking" before moving to a higher level. The "messing up" by dragging of Cabri drawings which are not based on valid geometrical constructions, provides students with this crisis of thinking. If they wish to produce a drag-resistant shape they must analyse which properties are to remain invariant and construct their figure accordingly. Students are thus forced into thinking about properties and relationships, facilitating their progression from Level 1 to Level 2 and in turn, from Level 2 to Level 3.

The study indicates that most students do in fact regress temporarily to the security of Level 1 visual construction when using Cabri. All but one of the students applied Level 1 by-eye construction methods, regardless of their initial measured level. Noss and Hoyles (1996) believe that these visual constructions are providing the essential scaffolding on which pupils can base their geometric constructions. While some pupils may be able to operate without by-eye constructions, others seem to require a visual construction before they can identify properties and relationships.

The study highlights the benefits of Cabri in developing pupils' confident use of geometric language, particularly if they are working in pairs where they can discuss their observations and construction strategies. Fundamental to this development, though, is the pupils' freedom to explore the Cabri tools. Once they discover the meaning through exploration of the terms in the menus, they are then able to incorporate these terms into their geometric vocabulary and employ them in their constructions.

Implications for teaching

It is important that Cabri tasks are structured carefully. Pupils first need to be familiar with the Creation items, since these are the tools which allow them to produce their visual shapes. The basic items of points, lines, line segments, triangles and circles can be explored creatively by children with little or no geometry experience. When names of shapes can be matched with a visual image, pupils can be introduced to the measurement of line segments and angles. Pupils at Level 1, such as the Group I students in this study, cope well with structured tasks involving drawing and measuring. Before angles can be measured, however, careful introduction to the various Cabri points is required. Unless pupils are aware of the status of the different points, *Basic point, Point on object, Midpoint* and *Intersection*, frustration and confusion will result.

It is the next stage, the construction of drag-resistant figures, which challenges students to think about properties and relationships. Such activities should be matched to the students' levels to avoid the inevitable confusion and inability to cope which appeared to be occurring when two Group I students, who were just at Level 2, were asked to construct the letter A shape. By gradually increasing the complexity of construction tasks, the teacher can develop the pupils' analytical skills and Level 3 understanding. Exploratory tasks, in which students are encouraged to experiment, conjecture and test and prove their conjectures, tasks which are eminently suited to the Cabri environment, will develop students' understanding to Level 3 and beyond. It is essential, though, that pupils who have just reached a level are given sufficient time to consolidate their understanding at that level.

The role of teacher intervention must also be considered. Hoyles and Noss (1996) noted that intervention involved them in a 'critical tension' due to the need

> to tread the line carefully between the pupils' room for manoeuvre and exploration on the one hand, and our own intentions and structuring on the other. We are convinced that without the former, much of the interesting potentiality of the computer is put at risk. Yet without the latter, our experience suggests that pupils' mathematical learning is at best haphazard (p. 32).

The students in this study required teacher intervention at times to assist them with choosing the appropriate Cabri points, but by carefully structuring the tasks so that they were appropriate to the students' levels, the need for intervention was kept to a minimum.

Any conclusions drawn from this research should take into account that the participants were generally highly motivated, with unrestricted access to the use of computers in Mathematics lessons. The study provides sufficient evidence, however, to justify further research on a larger scale to explore the full potential of dynamic geometry tools such as Cabri in the learning of geometry.

Acknowledgments

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What is 'meaning' in gender research?

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This is a paper about girls in mathematics. In it I give an account of a piece of research which I consider to be theoretically adequate for the work that I wanted to conduct about gender. It addresses the complex questions that people who do any research in mathematics education want to understand better: questions about learning and individuality, about thinking, about expertise. It links these with questions of power, control, dependence and change, and in so doing provides a different story about girls in mathematics.

Introduction

How do girls become gendered students in the secondary school mathematics classroom? Posing the research question in this form took some considerable time and a cursory reading of it gives no indication of the difficulty I had in formulating it. To arrive at this point I had to deal with questions at the theoretical level that concerned a basic assumption in mathematics education. I spent a long time trying to reconcile the theories of the gendered subject that mathematics education offered with what my own teaching experiences told me about girls in mathematics. In trying to unravel this knot of problems I was led to theories that I saw as more successfully able to account for what I saw happening in classrooms. These theories developed outside the field of mathematics education provided me with the conceptual resources to look at and think about the issue of girls in mathematics differently. What this entailed was questioning the essentialist girl of the gender project, something I was reluctant to do given its current status within mathematics education.

In the first section of the paper I look at the conditions which contributed to the way I carried out the research. I look at the assumptions about the gendered subject that presently underpin gender research in mathematics education. But because I wanted to provide a different story about girls in mathematics I journey outside binary logic and draw on ideas and concepts from recent social and political thought to develop understandings adequate for my purposes. This very situated piece of research about girls learning calculus is in a sense, then, a response to Leone Burton's (1995) challenge of doing research differently. In it I offer an alternative way of understanding subjectivity which can explain gender difference.

In the middle section I discuss the theoretical precepts which enabled me to carry out my work. I cite poststructuralist theories of language as providing the resources for this approach but in doing so I am mindful that those texts in this field which do engage, explicitly or implicitly, with this theoretical position are few in number. And those that do neither present a unified challenge nor universal approach to what constitutes gender research in mathematics education.

The final section is concerned with analysis. I analyse the accounts of experience given by participants and used as research data. What I am particularly interested in is the way in which students' use of language is linked to student meanings. I present and explain a theory of how meaning is achieved in the course of using it on my own material, centring the analysis around the concept of 'discourses' and students' positioning in relation to them. I discuss some of the methodological issues involved in the production and in the interpretation of accounts by telling the girls' stories, through different eyes from those that look through the lens of standard scientific methods.

The notion of the 'subject' in gender research

Gender inquiry in mathematics education has gained considerable rhetorical expedience both within and outside academic circles to such an extent that it threatens one with nothing more to say. Work by mathematics educators, both feminist and non-feminist, engaging with this issue has provided valuable understandings of the issues (Fennema, 1996) in which girls could be read not only as a fable of mathematics education but also as a means of transforming the school mathematics story. For the past thirty years the discussion has mainly been about images of girls in schooling and representation of young women in mathematics. More latterly the focus has included woman-centred research that privileges female difference. I will look at these two projects in turn.

The liberal feminist project, underlying what the gender/mathematics literature categorises as *Phase Three: Women as a problem in mathematics* (McIntosh, quoted in Rogers and Kaiser, 1995, p3) stresses the importance of equality of opportunity. Its perceptions of differences in the participation and achievement of girls from boys in school mathematics have generated an extensive body of research that maps the field, documents differences in learning style, achievement, or career choices, reports on school-based projects, seeks explanations, and offers compensatory measures to enhance girls' post-school options. Consequent to this work there has been an extensive engagement with the policy process the result of which has led to the implementation of a number of curricular and pedagogical programmes that are seen to promise improved emancipatory educational practice for girls.

These strategic projects are themselves an instantiation of the 'modernist' impulse. They accept uncritically a set of Enlightenment assumptions about individual freedom, critical reason and progress and their interventions become a means by which these ideals are substantiated and realised. These researchers assume one can specify a cause of oppression and have set themselves the task of helping girls to overcome this and to realise their full potential. The very rationale of their work is founded on certain understandings of the 'nature' and 'experience' of women and girls in society, and in their investments in the status of mathematics in society.

After this initial phase of uncovering gender blindness as well as the gender biases of the tradition, the task of feminist theoretical 'reconstruction' began. *Phase Four: Women as Central to Mathematics* focuses on women's concrete experiences across cultures, society and history and asks how the shift in perspective from men's to women's points of view might alter the fundamental categories, the methodology and the self-understandings of mathematics educational theory and practice. This work finds its strongest expression in feminist standpoints. This position is founded on absolute differences. It views women's creativity in mathematics as something different from male approaches to the point where there could be no overlap between the two. Research begins in questions arising from the perspective of women's activities and this perspective is seen as an important part of the data on which the evidence for all knowledge claims is based. Central to theoretical understandings in this feminist reconstruction of mathematics education are the categories of experience and feeling. Both experience and feeling are claimed as a kind of 'inner knowing', a heuristic of a deep truth, shaped by society but simultaneously the basis for oppositional schemes of truth.

What is important to my work is the status that the female subject embodies here in both these projects, and the possibilities that there might be for change. Within the framework of *Phase Three* the subject is seen as a highly rational, self-determining individual with an essential and relatively stable self. It is the individual of the humanist position:

> The humanist position tends to see the individual as the agent of all social phenomena and productions, including knowledge. The specific notion of the individual contained in this outlook is one of a unitary, essential non-contradictory and above all rational entity. It is the Cartesian subject in modern form: a notion of the subject

> which has been central to the whole of western philosophy founded on the principle of the cogito (Henriques et al., 1984, p93)

She takes the social structure as given but is able to exercise choice and self-determination to empower herself. Essentialism is her motivating text. At the core of her being is an essential self, yet she can liberate herself to achieve justice and independence. She is a reasoning subject capable of exercising individual agency; she says what she means and means what she says. Precisely because she is a biological female she has always been interpreted in gendered terms as dictating a certain identity, always situated in a world of universal roles and expectations. But as Walkerdine (1988) has ably pointed out, with the advent of new theories of the social her universality, her rationality, and her stability turns out to be contested and fictive.

The subject of the woman-centred approach is also circumscribed by essentialism, but for her the roles are reversed. For her, too, essence precedes her and is prior to the social field. Like the subject of *Phase Three* she is also a situated self, constituted by social roles, but for her, identity must be renegotiated. Her occlusion from the discourse of mathematics education is exposed and must be rectified by reversing the hierarchies, in which her experience as the source of knowledge and the focus of learning is central. Certain of her culturally specific qualities have their origins in biological difference and are 'naturally' female. Her feelings or emotions can be summonsed to test accepted claims of what is universally true .

Irigary (1985) has shown that the notion of woman's experience is subject to precisely the same limitations in understanding meaning as the notions that subject of humanism embeds. Underlying the idea of a unique experience of women is the seemingly unproblematic assumption that this experience can be identified and found to yield conclusions generalisable on the basis of gender. At the theoretical level there are difficulties in this claim because it rests on certain assumptions about commonalities in women's lives which have recently been challenged by postmodernism and by women of colour. Moreover, there are contradictions involved in claiming that experience is the source for knowledge while simultaneously arguing, as these theorists do, that these experiences are manipulated, shaped, and even oppressed by dominant discourses in society. What the standpoint 'subject' does in effect is reinstate the binary logic which it sought to deconstruct.

It is these very understandings which pose limitations on the kinds of questions that may be asked, and the kinds of research methods that might be utilised. Posing the question of gender in these terms binds gender research to the concept of binary opposition that is always inscribed in what Judith Butler (1990) calls a masculinist construction of an essentialized self. She argues that putting essentialism to strategic use presents some insurmountable problems for the development of alternative inquiry. The solution is not to replace it by a 'bigendered' conception of the individual in mathematics educational research, by bringing girls so-called 'ways of knowing' into the very definition of meaning. Such a solution remains indebted either to the confirmation or to the reversal of the politics of liberal feminism. It binds the possibilities of feminist research to the very problematic that it wants to challenge.

In no way do I wish to be seen as denying the personal and political importance of past and present feminist inquiry in mathematics education. But it is not good enough to refer unproblematically to experience. Emancipatory intentions and resolutions that are predicated on "essentialized, romanticized subjects" (Lather, p131) cannot engage with the complexity of the girls and mathematics issue. They are not able to consider the way power differentially constitutes people and empowers them to perform particular kinds of tasks, thus constructing specific kinds of subject. They are blind to the constraints of the material world and the various interactions of power and privilege that constitute social reality and in doing so their analytic value is reduced in explaining how subjects come to know.

Poststructuralism in gender research

If the essentializing manoeuvres of the contemporary woman-centred standpoint in this field have turned upon themselves into posturing the subject as fixed, stable and universal, how can we go beyond *Phase Four* politics? How is gender research into mathematics education to proceed when the general subject of essence turns out to be a questionable basis for investigation because it locks inquiry within a powerful fiction - a fiction that posits gendered identity as a construction of masculinist binary thought?

Poststructuralism is an intellectual movement that looks beyond these parameters and simple solutions. Poststructuralism, in developing Nietzche's ideas of the self as a willed unity masking a chaos of conflicting and contradictory desires, has provided a critique of the unitary individual by uncovering those meanings which are taken-for-granted in the term. This theoretical position transforms the essentialist theory of girls in mathematics education which insists on biological 'givens' as a source of meanings to a more situated theory of subjectivity. It refuses to fix meanings. It works against the stability of identities and experiences associated with current gender research in this field, by presenting a much more ambivalent and less fixed positioning of the subject.

Subjectivity is a central theoretical concept in poststructuralist theorising. In her book *Feminist Practice & Poststructuralist Theory* Chris Weedon (1987) explains the term in this way:

> ...the terms subject and subjectivity...make a crucial break with humanist conceptions of the individual which are still central to Western philosophy and political and social organization. 'Subjectivity' is used to refer to the conscious and unconscious thoughts and emotions of the individual, her sense of herself and her ways of understanding her relation to the world (p32).

For Michel Foucault the concept of 'discourse' is a key term in understanding subjectivity. Discourses are identified in poststructuralist thought as historically variable ways of specifying knowledge and truth. In other words they function as sets of rules, and constrain and enable what is possible to speak or write about, to think of, or imagine at a given moment. For Foucault discourses are not just representations of reality they actively produce it. They do this through language. Discourses are therefore powerful and provide an understanding of the relationships between subjectivity, language and social institutions. They provide a way of understanding meanings and possibilities of the subject.

Discourses form meanings and possibilities and take on and produce the subject and social change. Foucault argues that categories and concepts are never well-defined. They are implicated in what he calls knowledge/power. The 'girl' is a social self and this social self is *not* essential, but historically variable. I understand from this then that the category 'girl' has meaning only in the power of discursive truths which define and regulate it. To be categorised as 'girl' is to be implicated through discursive practices. The subject 'is' the discursive practices which produce her. She is also produced in power relations. She is constituted as a subject in discourses and practices and also, at the same time knowingly or not, contributes to the process of 'making' herself into particular kinds of subjects. Judith Butler has been at pains to clarify this constitutive relation: "There is no self...who maintains integrity prior to its entrance into this conflicted cultural field. There is only the taking up of tools where they lie, where the very 'taking up' is enabled by the tool lying there" (1990, p145).

Discursive practice is constitutive in both conventional and creative ways: it contributes to reproducing subject positions, yet is also contributes to transforming them. These transformations originate partly through constraints and possibilities for change found in the wider social structure and at all levels. The picture is one of changing configurations as discourses change. The 'girl' is subject to continuous transformations both between discursive formations and with a given discursive formation. She is an effect rather than the origin of particular, historically located practices. The girl in schooling might, on the one hand, be considered still a 'given' but, on the other hand, she is a 'given' produced in and by discourse. She can never be defined once and for all. She is always on the move, "dynamic and multiple, always positioned in relation to discourse and practices and produced by these" (Henriques et al., 1984, p3).

Her former essentialist identity gives way to a 'plural' and complexly constructed conception of social identity. Usher and Edwards (1994) put it like this: "the 'centred' subject does not exist naturally and pre-formed but is rather a cultural *construct,* inscribed by the meaning system that is language and by discourse, particular and systematic uses of language" (p16). The decentred subject, the 'girl' constructed by language, discourses, desire and the unconscious, has at last been shown to be an *unstable* category but one whose instabilities are of much interest to poststructuralism.

The Method and the Analysis

What is of major significance here for my work is the view of discourse as constitutive - as contributing to the production, transformation, and reproduction of the 'girl' of the mathematics classroom. It opens up a way for me to engage with the category 'girl' yet not be reduced to its essentialism. Rather than looking at 'the' girl in mathematics I want to give an account of how the girl becomes a gendered student in school mathematics. I will not be trying to discover or uncover one 'true self' of the girl but will come upon a multitude of selves found in the different kinds of linguistic practices that she articulates. It will not involve looking for the true nature of the girl in mathematics education but how she is continually being constituted through discourse and what the effects of this are.

In this work I wanted to consider the social positions the 'girl' could occupy and the functions of these positions in the diversity of discourses. I wanted to uncover they way in which she is caught up within discourses and power relations in school mathematics. There were many possible research methods which I could have made use of to do this but most methods would have undermined my original intentions by reducing the phenomena to measurement and/or by trivialising gender. They would not have dealt adequately with the problem of the essentialised subject.

Since I was committed to the application of poststructuralist ideas about the relation between meaning and subjectivity I chose a method that looked at discourse. My method was based on students' accounts of their experiences through their private talk as recorded on audio-tape throughout a series of lessons. I used this as my research data, together with my research journal, with their biographies, with photocopies of their work and with their private conversations with me.

I made use of a method of analysis that synthesises language and social theory. Discourse analysis draws together the work of Foucault and linguistics to consider how discourse constitutes social subjects and social relations. The method considers the role of language in the power relations and power struggles that shape and transform those discourses. It does this in a way that extends Foucault's conceptions of power and resistance which tend to ignore the way in which the subject is an *active* participant in power contestations. Discourse analysis looks instead at practice, at instances of what people actually do and say.

In the extract which follows, there are a multiplicity of discourses constituting the subject, expressed either implicitly or explicitly in the classroom. But my writing up of this transcript is, using Derrida's words, writing 'under erasure'. I am well aware that it is eclipsed by my understanding that gender is a social production. Its representation here, like all representation, is not a neutral process. This idea from poststructuralism suggests that my choice of text is contingent upon my theoretical interests. It also suggests that the meanings that I could generate from it are varied and multiple. No analysis that I could make of it could possibly expect to tell the 'truth'. The chosen text and the tasks I desire of it are fundamental to the questions surrounding my theoretical interests. In this way the extract becomes a production, not an already-there.

The class have been following their teacher through differentiation of functions with negative and fractional exponents. Rachel and Kate are seated in front of Richard and Blair.
(Simultaneous speech is shown as indented words on two or more lines. Pauses are shown as full stops. Capital letters show loud speech. An equality sign indicates one utterance following immediately on from another).

Mrs S: ...and we change that to negative four over three x, to the power of four thirds
Now this boils down to being able to handle exponents. OK? Any
Rachel: (*whispers*) I can't find it. Oh, shit! Bloody
Mrs S: questions? Just a matter of practising it.
Rachel: book!
(*to Kate, seated next to her*) How many books have they got up there?
Must be at least two.
Kate: (*whistles softly to get his attention*) Richard we need a book=
Rachel. =A BOOK.
(Blair is not willing to give Rachel and Kate a book)
Rachel: Isn't he MEAN.
(Richard gives his book to Rachel)
Rachel: Even on the right page I think.

It is Rachel who controls the interaction. She does this forcefully. The fact that she does so at all goes against any straightforward cultural myth of female passivity. What is at issue here is the way in which she is implicated in the powerful discourse of the educational process.

As one of the most powerful discourses in society education gives discursive meaning to its world and to its institutional practice by establishing the forms that it can take. Its very constitution as a discursive practice is founded specifically on the principles of progress, rationality and coherence, and these have fixed the possible meanings and functions of the teacher and the student. But this is done to such an extent that one tends to perceive these meanings as pre-formed, as 'ready-made', as natural.

The discourse of the educational process is powerful in the sense of normalising and regulating the practices of the student and those of the teacher that constitute them as particular kind of subjects. It does this through hierarchy, surveillance and regulation. Classroom practice plays a crucial role in its expression. The classroom becomes a site of normalising practices and positions the student as an object of disciplinary gaze of the teacher. It is here where discursive rules and relations are at work which regulate, make possible, and yet delimit who can speak, under which conditions, about what kinds of problems, and determines standards of acceptability. The authority that these rules and relations embody construct the versions of the student in the mathematics classroom that we know today. Yet this is not to claim that these normalising practices are necessarily a matter of overt control. Rather they work pervasively and often invisibly so that every student and every teacher becomes naturalised within the system.

Rachel like any other student has come to take them for granted. She has learned long ago the classroom's ways of doing things and which practices are valorised and which are not. Among other things she has learned to bring her books to class.

Later on in this lesson she says:
Rachel: But I don't know how to do number three.
Kate: Well ask. ASK.
Rachel: No. Cos I'm SCARED of that woman.

Power is exercised both through its visibility and its invisibility. Why does Rachel ask the boys for a book? Why not ask girls? There are other matters at work here. It is too simplistic to make a claim for role reversal. In our conversations she has told me that the four of them (Rachel, Kate, Blair, and Richard) have all been together in classes for two years and have all been promoted a class in mathematics. My research journal records the seating plans for each lesson and I note that they always sit near each other. Being together presents a confident and consolidated force to the other students in the class who are both older and unknown to them. Borrowing from the boys has a lot more to do with familiarity than subversion of gender roles.

Gender relations have historically structured women's absence from the active production of mathematical knowledge. At Rachel's school almost all of the mathematics teachers are women. As a group they claim an allegiance to liberal feminism. They stress the rights of female students to self-determination and their foremost objective is to create the material conditions at school so that this student potential might be realised.

As a female student in mathematics Rachel has a range of possibilities open to her. But these other possibilities involve accepting, negotiating or rejecting what is currently mediated through a variety of sources to which she has direct access: movies, television, advertising, the magazine which she buys *(Girlfriend)*, her family, her involvement in Girls Brigade, and her friends.

All of these construct gendered subject positions in their own particular ways. They are not all compatible. Rachel has learned to make choices between them.
She expresses her choice through actively participating in the creation of knowledge in the following extract, taken from the same lesson :

Rachel: So, number three. Two over square root of x.
What?
Two square root of x. I don't know how to do this
(to Richard, seated behind her) How do you do this?
How do you do the square root ones?
Didn't mean that to be RUDE right?
(classroom becomes momentarily quiet)
Shhhhit! SO EVERYONE GOES QUIET
So what do you do?
Richard: I don't know.
Rachel: (to Kate) How do you do number, oh, number three?
What's the answer for number three?
Kate: You've got the wrong number. Isn't it one over x?
How come you've got two over x?
Rachel: Should watch what I'm doing.
You go one over square x or something like that.
Kate: One over x to the half. The square root it's a half, isn't it?
Oh. I don't know.
Yea. It's the square root.
Rachel: I don't know
(sounding cheerful) Well, try it, TRY it.
Pick a number and square root it and then pick a number and put it to the power of a half and see if you get the same answer.
OK. Well I'LL DO IT THEN.
Just pretend to be useful.
Root nine equals. OK. Nine power one over two . YEP!
You're right! Beautiful!

What does being a student mean to Rachel? For that matter, what does being a student mean to any of the students. All have different meanings and each one is positioned in relation to these meanings. Each meaning is produced for a specific time and within particular relations and discourses. These meanings have much to do with personal history, culture, unconscious processes and social differences. But above all they have a lot to do with networks of power. If this is so, then meanings that people produce not only maintain these subjectivities they also have the capacity to transform them.

Conclusion

This piece of research, in both the method that it describes and the theoretical assumptions it presumes, makes a beginning step towards a theory and practice capable of grasping the complexities of what it means to be a gendered student in the mathematics classroom. It marks a theoretical move outside the binary logics of increasingly inadequate category systems into the world of the mathematics student. It also represents my attempt to analyse some of tensions, confusions and dilemmas that make this world so seemingly problematic. Through its production I have come to an understanding of students' classroom experience not as orderly and organised as often portrayed in the research literature, but lived as contradictions, conflicts and comprises.

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Students' understanding of the concept of a variable

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This paper explores the use of written tests and semi-structured interviews in ascertaining students' understanding of the concept of a variable. A written algebra test was administered to 379 students and from the results students were selected for a semi-structured interview. The types of questions asked and the medium in which they were asked appeared to influence the responses given. It is conjectured that these issues must be considered when endeavouring to reach a richer understanding of the students' perception of the concept of a variable.

Algebra has long held a place of distinction in the mathematics curriculum (NCTM, 1989; National Statement for Australian Schools, 1991). Few have contested the importance of algebra as it is seen as 'the language through which most mathematics is communicated' (NCTM, 1989, p. 150). Critical to the algebraic domain is the variable construct. The literature identifies distinct stages students pass through when reaching an understanding of the concept of a variable.

Stages of understanding the concept of a variable

Through his large-scale study of students' interpretations of literal terms, Kuchemann (1978, 1981) identified six different stages. These were based on stages developed by Collis (1975) and are as follows:

Letter evaluated: Students assign numerical values to letters at the outset of a problem. For example, when asked to describe the expression $2+3x$ children often assign a value to x , such as 1, and compute the answer. Thus $2+3x=2+3\times 1=5$.

Letter not used: Here, students ignore the letters, or at best acknowledge their existence but without giving them meaning. For example, the algebraic expression $2x+8y+3x$ is equated to $13xy$. Such an answer is obtained by simply adding up all the numbers, then writing down each letter that occurs.

Letters used as objects: Here, students regard the letter as shorthand for an object or as an object in its own right. For example, $2a+3b$ represents adding 2 apples to 3 bananas. This is referred to in the literature as 'fruit salad' algebra (Booth, 1988).

Letters used as a specific unknown or constant: Students perceive the letter as a specific but unknown number. For example, the expression $\mathrm{L+M+N}$ would never equal $\mathrm{L+P+N}$ as N cannot equal P. Even though both N and P are acknowledged as variables they must always be different values from each other as they are represented by different letters of the alphabet.

Letter used as a generalised number: Here, students perceive the letter as representing, or at least as being able to take, several values rather than just one. For example, if students are asked to list all the possible values for the expression $x+y=10$, they will list more than one of the whole numbers which will satisfy the condition. However, they tend not to realise that all numbers that satisfy the condition are required.

Letter used as a variable: In this final stage, students see letters as representing a range of unspecified values including all the rational and irrational numbers. Students also understand that the variable is defined by its relation to other terms in the expression. For example, in the expression $x+y=10$ the value of x is dependent upon the value of y.

Kuchemann (1981) found that most students (aged 13-15 years) could not cope consistently with items that required the use of the letter as a specific variable. Kuchemann (1981) stated that generally the first three stages indicated a low level of response.

Common errors and misconceptions of the concept of a variable

The first five stages delineated by Kuchemann (1981) also represent common misconceptions children experience in interpreting algebraic symbols. For example, when examining algebraic expressions, students may assign a numerical value to the variable, ignore the variable, view the variable as a specific unknown, or see the variable only consisting of whole numbers. In addition to the above misconceptions a further three are identified in the literature. These are:- first, changing the variable symbol as changing the referent. That is, different variables must take on different values. For example, the expression $3n$ is not the same concept as $3x$ as n and x could never be equal (Booth, 1984; Chalouh & Herscovics, 1983; Wagner & Parker, 1993). Second, assigning numerical values to letters according to their rank in the alphabet, for example, $a=1$ and $z=26$, or if $x=3$ then $y=4$ and $z=5$. (Booth, 1984; Chalouh & Herscovics, 1983). Third, assigning the letter as a subdivisional label, for example, 3a refers to the first part of the problem (Chalouh & Herscovics, 1983).

Most of these misconceptions were delineated from the results of large studies involving written tests. The types of questions asked and the medium in which they were asked could influence the responses given. Through written responses students may not be able to exhibit their full understanding or misunderstanding of the concept of a variable. This paper compares the results from written responses and a semi-structured interview and explores how each contributes to reaching a richer understanding of students' perceptions of the concept of a variable.

Methodology

This study comprised whole class testing to select students for in-depth interviews. The sample for the whole class testing comprised 379 students from one independent coeducational school and one state coeducational school in the metropolitan area. The students' ages ranged between 12 years and 2 months and 15 years and 10 months. Both schools chosen for the study consisted of students from lower-middle socio-economic status, with a variety of ethnic backgrounds represented. Both schools were large metropolitan schools comprising up to two thousand students.

Whole class test: The whole class test was administered as a written test and completed within a one-hour period. The test consisted of three components, namely, generalising from visual patterns, generalising from tables of data, and understanding the concept of a variable. The results from 3 of the 8 questions selected for the variable component of the written test are discussed in this paper.

Semi-structured interview: The results for each component were ranked and three students were selected at the 1st, 25th, 75th and 100th percentile for each of the written components. These particular percentiles were chosen to highlight the difference in levels of understanding of the concept or a variable. Each interview was videotaped and transcribed.

Instrument

From the results of the written test 2 questions were selected for inclusion in the semi-structured interview, and a third question was adapted for the semi-structured interview. These 3 questions were believed to probe students' view of the variable, their need for closure of algebraic expressions, their propensity to concatenate algebraic expressions, and why students allowed "= constant" to limit possible values for the variable. The selected questions were:

Question 1. **This question is about $t+t$ and $t+4$.**

(a) **Is $t+t$ ever larger than $t+4$? If so when?**

(b) **Is $t+4$ ever larger than $t+t$? If so when?**

(c) **Are $t+t$ and $t+4$ ever equal? If so when?** (Modified Harper, 1979)

This item was modeled on an item used by Harper (1979) and Quinlan (1992). The item used in their studies was as follows:

This is a question about $t+t$ and $t+4$
(a) Which is larger, $t+t$ or $t+4$? WHY? *(b) When is $t+t$ larger?*
(c) When is $t+4$ larger? *(d) When are they equal?*

In the interview stage of the pilot study, concern was expressed about the inconsistency in the wording of the Harper item. For example, *"When is $t+t$ larger?"* seemed to alert to the fact that perhaps $t+t$ could be larger. This proved confusing for students who had chosen $t+4$ for part (a). Hence the question was reworded so that such inconsistencies no longer existed.

Question 2. **For a school excursion, 3 buses take f students each and 4 cars take g student each.**
(a) **Give the total number of students taken by these buses and cars.**
(b) **One car leaves early with g students. How many students remain?** (Quinlan, 1992b)

This question was created to obtain information on students' ability to interpret the meaning of letters when they referred to a real-life context and to carry out operations on numerical variables without knowing their values.

Quest 3. (a) **If $c+d=10$, tick ALL the meanings that c could have:**
3 10 12 7.4 the number of apples in box
an object like a cabbage an object like an orange
(b) **If $c+d=10$, what happens to d as c gets bigger?**
(c) **If $c+d=10$, and c is always less than d, what values may c have**(Harper, 1979.Kuchemann, 1980 & Quinlan, 1992b

The origin of this item is threefold. It is based on the CSMS project question (Hart, 1981) *"What can you say about c if $c+d=10$ and c is less than d,"* (Kuchemann 1980, p. 67) and on Harper's (1979) equation task, *"If $x+y=10$ when is x less than y."* Kuchemann (1980) regarded this question as one which required students to view letters as generalised numbers. Quinlan (1992, p. 106) added parts (a) and (b) to the original question. Part (a), measured the types of possible meanings students are prepared to accept for alphabetic symbols in algebra. Part (b) tested students' understanding of the covarying relationship between the two variables (i.e., as the value of c varies so does the value of d). In the semi-structured interview students were asked *"If $p+m=12$, put a ring around all the possible values p can have. 4, 12, 15, 0, 3.9, -2, an object like a pear, the number of children in the class."* Students were then asked to articulate their reasons for including or excluding certain values.

Results

Written component

Question 1.: For each part of question 1, students were required to give two responses. Students were firstly asked to reply either in the affirmative or negative to the question and then were asked to elaborate on their response with an appropriate reason. Table 1 summarises the results for this question.

Table 1
Percentage of Students who correctly responded to Each Part of Question 5

Question 1	Correct response	Valid reason
(a) Is t+t ever larger than t+4? If so when?	46.4	28.8
(b) Is t+4 ever larger than t+t? If so when?	45.9	29.0
(c) Are t+t and t+4 ever equal? If so when?	43.0	36.1

Less than half the students answered this question. The results seem to indicate that giving a valid reason for the equality situation was easier than the other two situations. The rewording of the Harper Quinlan question appeared to result in a greater percentage of students reaching correct solutions. While Quinlan (1992) found that 50% of his sample (n=517) responded correctly, 47% (n=241) of those who offered a correct response had completed at least 2 more years of schooling with 21% being in their final year. The students' responses to question 1 were also assigned a level according to their understanding of the relationship between $t+t$ and $t+4$. Each level represents a developing understanding of the concept of a variable. Table 2 describes each level with the percentage of students whose responses were allocated to that particular level.

Table 2

Levels of Responses for Question 1

Level	Description	% response
0	Did not answer	56.2
1	Letters represented objects	0.5
2	Focus on the constant rather than the variable	4.2
3	Use of only one trial example to reach a solution	4.7
4	Correct response for specific values	5.0
5	Correct response in general terms	29.3

The majority of students did not respond to this question. It appeared that only 29.3% of the students saw the variable in general terms.

Question 2: Responses to both part (a) and part (b) seemed to fall into four distinct levels Table 3 presents a summary of a description of the levels together with the percentage of students whose responses were considered to be at that level.

Table 3

Levels of responses to Question 2

Level	Description	% response	
		2 (a)	2(b)
0	Did not attempt or assigned a number	29.0	31.4
1	Incorrect algebraic response	14.5	21.4
2	Correct response but the need for closure	7.7	10.8
3	Correct response	48.5	36.4

Students found part (a) easier to respond to than part (b). This is not surprising given that a correct response to (b) appears to be contingent on a correct response to (a). Also the need for closure was more prevalent for (b).

Question 3: Table 4 summarises the percentage of students who chose the various options in part (a) of question 3.

Table 4

Percentage of Responses to each Choice for Question 3(a)

	Choice						
	3	10	12	7.4	number in box	object cabbage	object orange
accepted	76.5	51.5	11.9	66.5	40.4	22.7	21.4

The majority of students accepted 3 and 7.4 as possible values for the variable c but were less confident in allowing c to be 10. When c is 10 the value of d is zero. Only eleven percent accepted c as 12. When c is 12 the value of d is negative. These findings seem to indicate that the nature of the product, "equal ten," of the equation plays a significant role in limiting the permissible values for the associated variables. This trend was difficult to delineate in the Kuchemann (1980) format of the question. For the majority of students, it seems that '=10' represents a barrier beyond which values cannot be selected. This could reflect reluctance by the students to assign zero or negative values to the variable *d*. This issue was investigated in the semi-structured interview. Only 40.4 percent of the students accepted that *c* could be the number of apples in a box, whereas up to 22.7 percent allowed *c* to stand for an object such as an orange or cabbage. This seems to indicate that the students had difficulty in accepting the variable as a generalised number and in differentiating between this interpretation and the variable as an object.

Levels were also developed for classifying responses to part (b) and part (c). For part (b), the levels indicated whether the students could see the covarying relationship between *c* and *d* in the equation $c+d=10$. At the lowest level students failed to see the interrelationship between *c* and *d*. For part (c), the levels indicated the possible values students would accept for *c* if *c* is always less than *d* in the equation $c+d=10$. A level 4 response was '<5' and a level 5 response was '<5' but also included 0, negatives and fractions. This distinction was made as it was felt that some student's response of '<5' might only consist of the positive integers less than 5. The following tables describe the levels and record the percentage of students whose responses were classified at that particular level.

Table 5

Levels of Responses for Question 3(b) "If c+d=10, what happens to d as c gets bigger"?

Level	Description	% response
0	Did not respond	20.1
1	No change (Fails to see the interdependence between the 2 variables)	6.3
2	Bigger (sees a relationship between the 2 variables; incorrect response)	5.3
3	Smaller (i.e., as c gets bigger d gets smaller)	68.3

Table 5 indicates that 68.3% of the students reasoned correctly that *d* would get smaller.

Table 6

Levels of Responses for Question 3(c) "If c+d=10, and c is always less than d, what values may c have?"

Level	Description	% response
0	omitted response or incorrect list of numbers e.g., 1,2,3,4,5,..10	29.4
1	one or two numbers as response e.g., 3,4	4.5
2	1,2,3,4 – included only integer responses	42.9
3	included 0, negatives and/or fractions (4 & under or 4.99 down to 0)	20.6
4	< 5	2.1
5	< 5 and included 0, negatives & fractions	0.05

As can be seen from the above results very few students responded correctly to question 3(c). Over half of the students tested gave at least four values for *c*. Kuchemann (1981, p. 108) found that nearly 40% of his sample of 14 year old students (n=1000) gave only one value for *c* and that 30% gave four or more. For this study, approximately 43% of the students gave four values for *c*. As indicated by Quinlan (1992), it appears that the inclusion of the first two components helps students frame appropriate answers to (c).

In summary these 3 questions from the written test indicated that when examining the relationship between two variables in an expression of the form $x+y=c$, where c is a constant, the value of the constant seems to play an important role in the permissible values students will accept for the variables themselves. It appears that the constant, c, represents a barrier. Students exhibited a reluctance to choose values for either x or y that were greater than c. Even students who were considered by their teacher to be very capable of understanding algebraic concepts, included the letter standing for an object in a number of their responses. A very small percentage of the students perceived the variable as a generalised number. Most students tended to see the variable as being represented by up to 4 specific integer values. These trends were investigated further in the semi-structured interview.

Semi-structured interview

Twelve students were chosen to be interviewed on their understanding of the concept of a variable. From the literature it seems that students' perceptions of the concept of a variable range from the letter standing for an object through to the variable as a generalised number. The semi-structured interview attempted to explore some of these understandings. The following section presents a summary of students' perceptions.

Variable as an object: The results for question 3 from the written algebra test indicated that 22.7% of students allowed c to stand for 'an object like a cabbage' and 21.4% allowed c to stand for 'an object like a orange' (see Table 4). From an analysis of the written responses, some of the more able students seemed to hold this misconception. The inclusion of "object like a pear" in the third interview question probed why students accepted the variable as standing for an object. Not one of the twelve students interviewed accepted p as "shorthand for pear." When questioned, those students who ringed 'an object like a pear' seemed to assign a number to p. For example, Jason at the 1st percentile said, "Yes it could be about the number of pears and number of apples." Michelle, also at the 1st percentile said, "Could be 14 pears." One student at the 100th percentile could clearly articulate why 'an object like a pear' was not a valid option. Mary said, "Object like a pear and 12 have no relationship," indicating that she viewed 'pear' as an object and not as a number. The two other students at the 100th percentile said. "p could be an object like a pear only if m was 11 pears and then you would have 12 pears." The interview seemed to indicate that, contrary to the belief held when analysing the written component, simply ringing 'object like a pear' did not necessarily mean students saw p in expressions like $p+m$ standing for pear, but rather as 'a pear', that is, 'one pear'. When questioned, students who ringed this option, expressed valid reasons for its inclusion in their options. This seems to indicate that relying solely on written responses could lead to erroneous conclusions with respect to students' understanding of the concept of a variable.

Specific unknowns: From the students' responses there seemed to be a growth of acceptable values for the variable. Students at the 1st and 25th percentile made decisions about the two expressions $t+t$ and $t+4$ using one specific value for the variable. Adam said, "$t+t$ is bigger when $t=5$, $t+4$ is bigger when $t=2$ and they are the same when t equals 4." By contrast, one student at the 75th percentile reached a conclusion by using a series of specific unknowns for t. For example, Jenny said, "$t+t$ is bigger when t is 5 or 6.,$t+4$ is bigger when t is 1,2, or 3." For this problem, the other two students at the 75th percentile perceived the variable as a generalised number, that is, "$t+t$ is bigger when $t > 4$ and $t+4$ is bigger when $t < 4$".

Generalised number: For the expressions $t+t$ and $t+4$, two students at the 75th percentile and three students at the 100th percentile immediately articulated the general case, that is, "$t+t$ is larger when t is greater than 4, $t+4$ is larger when t is less than 4 and they are equal when $t=4$." It was conjectured that they had an understanding of the variable as a generalised number, that is, thinking at level 5 (see Table 2). Students at the 1st and 25th percentiles either could not solve the problem nor needed specific values in order to reach a solution.

Closure: For question 2, 3 students at the 1st and 25th percentile wrote $c+f=$, indicating a need for closure. All the students at the 75th and 100th percentile did not need to close the expression, writing $c+f$.

Product component of an equation: For the expressions $p+m=12$ it seemed that the '=12' component of the question was not the only factor that limited the number of acceptable values for p. The addition sign before m also seemed to play an important role. The six students at the 1st and 25th percentile failed to accept 15 for the value of p. A common response was "15 plus anything is over 12," indicating that the value of m must be positive as it was preceded by '+'. Of particular interest was the fact that they all accepted '-2' as a value for p. One student stated that, "-2 was alright as it is below 12."The other six students at the 75th and 100th percentile accepted 15 as a valid value for p, stating that m would need to be negative.

Comparing $t+t$ with $t+4$ by considering the constant only: Only two students seemed to compare $t+t$ with $t+4$ by focussing on the value of the constant. Lachlan, at the 25th percentile said, "$t+4$ would always be bigger because the 4 made it bigger." Lindsay, at the 100th percentile, also seemed to allow the value of the constant to influence his decisions. He said, "for $t+4$ you add 4 to a variable whereas $t+t$ has two variables so it must always be bigger," suggesting that an expression containing two variables will always be bigger than an expression only containing one variable. Adam, at the 25th percentile, continually asked if the t's were all the same. He seemed to want to substitute different values for t in the two expressions.

Concatenation of algebraic expressions: Three students exhibited a need to concatenate the variables. All of these students were either at the 1st or 25th percentile. For question 1, Chris at the 1st and James at the 25th percentile indicated that they would have 'cf trucks'. When comparing $t+t$ and $t+4$, Chris also suggested that, "$t+t$ is $2t$ and $t+4$ is $4t$ so $t+4$ would be bigger."

In summary the results of the interviews indicated:-Students at the 1st and 25th percentile were characterised by their need to evaluate, close and concatenate algebraic expressions, and by perceiving the variable as representing a specific unknown. Two students at the 75th percentile and the three students at the 100th percentile perceived the variable as a generalised number and did not concatenate or close any of the algebraic expressions.

Discussion and conclusion

Both the results of the written test and the interview support Kucheman's stages of understanding of the concept of a variable. Not only did the less able students, students at the 1st and 25th percentile, seemed to perceive the variable as representing a specific unknown but they also exhibited a need to close algebraic expressions. By contrast, the more able students did not need to close algebraic expressions and they perceived the variable as a generalised number.

The results also seem to indicate that care must be taken when conjecturing about students understanding from written responses. First, the framing of the question itself could limit the types of responses given. The use of appropriate prompts could help elicit students' full understanding of the concept of a variable. For example, as indicated by Question 3 in the written test, the inclusion of (a) and (b) helped students frame a response to (c), and for question 1 the rewording resulted in a greater percentage of students reaching correct solutions.

Second, the medium in which questions are posed needs also to be considered when reaching conclusions from students' responses. From an analysis of the results of the written algebra test it seemed that many students perceived the variable as a letter standing for an object. The results of the interview seemed to indicate that this assumption was too simplistic. None of the students interviewed believed that p was "shorthand for pear." All the students who selected p as standing for an object like a pear, had valid reasons for its inclusion and these reasons relied on p standing for a number of pears and not pear. Also, for the expressions $p+m=12$ the product component (=12) is not the only factor that limited students' thinking. It seemed that the placement of '+' before the variable also played a key role. Students at the 1st and 25th percentile believed that '$+m$' meant that the value of m must always be positive. This seems to indicate that some students are not capable of delineating between '+' as the operation of addition and '+' indicating a positive value. Both of these misunderstandings are difficult to identify in a written test format.

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Numeracy Benchmarks for Years 3 and 5: What about Chance and Data?

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Since January 1997 there has been much debate within Australia on the type of numeracy benchmarks which should apply to children in Years 3 and 5. As well as debate on the breadth and depth of understanding, there has been difficulty in some areas establishing what children actually do know and can do. Although the data included in this report were not collected to answer questions related to numeracy benchmarking, they may help inform the debate about what children in Years 3 and 5 know and can do in the area of chance and data.

Numeracy benchmarks for Year 3 and 5 children in Australia have been under consideration since January 1997 under the auspices of the Task Force on Literacy and Numeracy set up by the Australian Ministerial Council on Education, Employment, Training and Youth Affairs. With the advice of academics from around the country and work of people employed by the Curriculum Corporation, the following definition of Numeracy was the agreed foundation from which the benchmarks would be developed.

> Numeracy is the effective use of mathematics to meet the general demands of life at school and at home, in paid work, and for participation in community and civic life. For the purposes of this project, the numeracy benchmarks ... will incorporate the development of students' understanding and competence with number and quantity (ie, measurement), shape and location, and the handling and interpretation of quantitative data. (K. McLean, personal communication, March 3, 1997)

The debate that has surrounded the development of the benchmarks has variously related to the breadth of application involved in using numeracy skills, the depth of conceptual understanding, the need for agreement among all states on each detail included, and the lack of data available to back up claims of what Year 3 and 5 students understand and can do. This paper seeks to address only one issue. It will examine responses from Year 3 and 5 students to nine items in the area of chance and data (Moritz, Watson, & Pereira-Mendoza, 1996; Watson, Collis & Moritz, 1994, 1995, 1997). It is hoped that such information can provide a starting point, both for further debate about numeracy benchmarks and for following the progress of students as chance and data take a more prominent place in the Australian mathematics curricula.

Method

The data reported here were not collected to answer specific questions related to numeracy benchmarking in Australia. They were collected to document the cognitive development of students across the years of schooling, initially in 1993 from Years 3, 6 and 9, and to evaluate the implementation of the chance and data curriculum in Tasmania from that year. To that end further data were collected from the 1993 cohort in 1995 and 1997, with new students introduced to the study each year. The sample reported here is made up of children selected from eight state primary schools in the seven regions of the Tasmanian government school system. The schools represented rural, urban and semi-urban areas and were either primary or district high schools. Either all Year 3 (and later Year 5) students from the school completed the survey, or the classes that participated were chosen by the school rather than the researchers. For the purpose of this report, data from 864 Year 3 students in 1993, 1995 and 1997 have been combined, as have data from 703 Year 5 students in 1995 and 1997.

The nine items for which data are presented in this report (see Figure 1), were part of a 20-item chance and data survey (Watson, 1994) administered to students in class over 45 minutes. Year 3 students answered the first ten items and hence time was not a factor in the quality of responses. Where necessary students were read the items but no further help was provided. Item 1 of the survey, which asked for events that were certain, impossible or possible is not discussed here due to space restrictions (see however, Watson, Collis & Moritz, 1993; Moritz et al., 1996).

Q2. If someone said you were "average", what would it mean?

Q3. What things happen in a "random" way?

Q4. If you were given a "sample", what would you have?

Q5. Every morning, James gets out on the left side of the bed. He says that this increases his chance of getting good marks. What do you think?

Q6. One day, Claire won Tattslotto with the numbers 1; 7; 13; 21; 22; 36. So she said she would always play the same group of numbers, because they were lucky. What do you think about this?

Q7. Consider rolling one six-sided die. Is it easier to throw
☐ (1) a one, or
☐ (6) a six, or
☐ (=) are both a one and a six equally easy to throw?
Please explain your answer.

Q8. A mathematics class has 13 boys and 16 girls in it. Each pupil's name is written on a piece of paper. All the names are put in a hat. The teacher picks out one name without looking.

Is it more likely that
☐ (b) the name is a boy, or
☐ (g) the name is a girl, or
☐ (=) are both a girl and a boy equally likely?
Please explain your answer.

Q9. Box A and Box B are filled with red and blue marbles as follows:

Box A	Box B
6 red	60 red
4 blue	40 blue

Each box is shaken. You want to get a blue marble, but you are only allowed to pick out one marble without looking.
Which box should you choose?
☐ (A) Box A (with 6 red and 4 blue).
☐ (B) Box B (with 60 red and 40 blue).
☐ (=) It doesn't matter.
Please explain your answer.

Q10. A primary school had a sports day where every child could choose a sport to play. Here is what they chose:

	Netball	Soccer	Tennis	Swimming
Girls	30	5	15	10
Boys	0	20	18	20

(a) How many girls chose tennis? ____________________
(b) How many boys chose netball? ____________________
(c) How many children chose swimming? ____________________
(d) In which sport were boys and girls most evenly divided? __________
(e) Were there more girls or more boys at the sports day? __________
How do you know? ____________________

Figure 1. Items from the chance and data survey.

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The previous analysis of the 1993 data set of which some of these data are a subset was based on the Structure of Observed Learning Outcomes (SOLO) model (Biggs & Collis, 1982; Collis & Biggs, 1991). Detailed discussion of the model and its application to topics in chance and data is found in Watson et al. (1994, 1995, 1997) and Moritz et al. (1996). The requirements of a benchmarking exercise are consistent with the reporting of unistructural-multistructural-relational (U-M-R) levels of structural performance and hence the same model will be used here. The levels of response are the following.

IK: These responses are in the ikonic mode which is prestructural to the concrete symbolic mode, usually associated with imagination or tautology.

U_1 or U_2: Unistructural responses in the concrete symbolic mode employ a single element of the mode associated with the task set. U_1 responses are in the first cycle of the concrete symbolic mode, while U_2 responses are in the second and represent the consolidation of a first cycle concept which may have been structurally complex in its construction there but later is employed as a single element in more complex arguments.

M_1 or M_2: Multistructural responses in the concrete symbolic mode employ several elements of the mode associated with the task set, usually in a sequence. The distinction between M_1 and M_2 is again associated with the first or second cycle of processing within the mode and the nature of the elements used.

R_1 or R_2: Relational responses in the concrete symbolic mode relate together the elements of the mode in a fashion which is structurally consistent with the expectations of the first or second cycle and the nature of the task set. In the first cycle R_1 response are likely to be associated with the construction of a concept, while in the second cycle they are likely to be associated with the application of the concept in a more complex problem solving situation.

For purposes of statistical comparison, response levels were given integer values: 0 for IK, 1 for U_1, 2 for M_1, 3 for R_1, 4 for U_2, 5 for M_2, and 6 for R_2.

Of the Year 3 students, 560 were in the Year 5 group two years later and hence 243 of the Year 5 students had not been surveyed previously. These students were compared for each item with those who had been surveyed previously and it was found that for only one item (Q2) was there some evidence ($p = .04$) that students in Year 5 repeating the item did better than those who had not seen it before. Hence there was no overall effect which could be attributed to recall of the survey items. Cross-cohort analyses showed an improved performance for Year 3 students in 1997 on items Q2, Q3, Q4, and the last two parts of Q10, while Year 3 students in 1995 performed less well on Q8. The large sample sizes in the study meant that these differences amounted to a maximum difference of .2 of a SOLO level. Hence it was felt that the overall picture of performance of students would not be unduly biased by combining the data as done here. While the issue of this sample representing children in the rest of Australia is not simple, it should be noted that most other states and territories were implementing a similar curriculum from 1993-4 onwards.

Results

Language of Statistical Understanding (Items Q2, Q3 and Q4)

By middle primary school, students have often been exposed to chance and statistical terms which may provide an important foundation for later understandings. The following examples illustrate levels of student understanding for the terms "average", "random" and "sample" (see Figure 1, Q2, Q3 and Q4), according to the categorisation reported by Moritz et al. (1996). Some responses did not offer any appropriate meaning or example of the term, confusing it with other words.

P: [Random]: People kidnap people and demand some money. [Year 5]

Some responses indicated the student had heard the word in conversation, by offering examples of usage of the word, sometimes apparently without appreciating the meaning. Such responses were classified as ikonic (IK) for Q2 which asked for meaning, as an example is not in the required mode for the task set. For Q3 and Q4, the phrasing of the question meant that appropriate examples were classified as unistructural (U_1). Common examples for "average" included height and intelligence. Examples for "random" included natural unpredictable phenomena, a specific humanly constructed event or process, or games and competitions. For "sample", examples were foods and products, and natural samples in science or social science contexts.

IK [Average]: You are the average height of tallness. [Year 5]
U_1 [Random]: Wind. Rain. [Year 5]
U_1 [Random]: Like random breath test. [Year 3]
U_1 [Random]: Tattslotto balls. [Year 5]
U_1 [Sample]: Like a sample of urine. [Year 3]

Some responses used common, natural language to offer a single description related to the term used, and were also classified at the U_1 level.

U_1 [Average]: You are o.k. [Year 3]
U_1 [Average]: Normal. [Year 3]
U_1 [Average]: I am the same as everyone else. [Year 5]
U_1 [Random]: In no particular order. [Year 5]
U_1 [Random]: Not planned things. [Year 5]
U_1 [Sample]: A small something. [Year 5]
U_1 [Sample]: A demonstration. [Year 3]
U_1 [Sample]: Something the same as something else. [Year 5]

At the multistructural level (M_1), responses elaborated beyond the single general idea to describe multiple facets of the term. For "average", responses described aspects of how the average relates to a data set or to a method for obtaining the average from the data set. For "random", responses provided multiple examples from different contexts or an example with a simple defining characteristic. For "sample", responses added a complement to the U_1 concept, sometimes incorporating a concrete example.

M_1 [Average]: That I'm doing the same as most people in the class. [Year 5]
M_1 [Average]: That you weren't smart and you weren't dumb you're in between. [Year 5]
M_1 [Random]: Winning a lottery, eclipses. [Year 5]
M_1 [Random]: The weather is random, we cannot control it. [Year 5]
M_1 [Sample]: I'd have, if it was for clothes, a small piece of material. [Year 5]

For "average" and "sample" at the M_1 level the realisation of the part-whole nature of these concepts began to emerge. Compare for example, the third U_1 [Average] response with the first M_1 [Average] response.

Responses at the relational level (R_1) were rarely observed in responses by primary school students to these questions. This level of response demonstrates the acquired concrete symbolic concept of "average" as a representative measure of a data set, "sample" as a small part representing a whole, and "random" as unpredictable or uncontrolled with respect to natural events, while also possibly meaning unbiased or equally likely when concerning selection in games, competitions, or surveys.

R_1 [Sample]: A bit of something to show you what the whole thing is like. [Year 5]

The percentages of responses at each SOLO level are shown in Table 1, where it can be seen that by Year 5, 87% could at least give an example of an average and 76% had some appropriate idea about the word "sample", while only 35% were aware of the idea of "random". The corresponding percentages for Year 3 were 54%, 52%, and 9%.

Table 1

Percentage of Responses by Year to Items about Statistical Language

SOLO Level	Q2 "Average" Year 3	Q2 "Average" Year 5	Q3 "Random" Year 3	Q3 "Random" Year 5	Q4 "Sample" Year 3	Q4 "Sample" Year 5
NR/ P- Irrelevant	46	13	91	65	48	24
IK/U_1 - Example	17	10	5	16	17	15
U_1 - Concept	32	56	3	14	15	17
M_1	5	22	1	5	17	36
R_1	-	-	-	-	3	8
N	864	703	864	703	864	703

Beliefs about Luck (Items Q5 and Q6)

Students often hold beliefs about luck from an early age, and the difficulties of removing deeply held intuitions are commonly reported in the statistics education research literature. Levels of response to the items about luck (see Figure 1, Q5 and Q6), described by Watson et al. (1995), range from belief in luck, to uses of common phrases to express beliefs in negative recency of outcomes or equal likelihood. Some responses expressed agreement with belief about luck. These were classified as ikonic, i.e., prestructural to a concrete symbolic construction of situations involving chance.

IK [Bed]: It would help him get a good mark. [Year 3]

IK [Tattslotto]: They probably are lucky numbers. [Year 5]

At the U_1 level, responses offered disagreement to belief in luck.

U_1 [Bed]: He's just being superstitious. [Year 5]

U_1 [Tattslotto]: No I don't think they are lucky numbers. [Year 3]

U_1 [Tattslotto]: Wrong, lightning doesn't strike in the same place twice. [Year 3]

To the Tattslotto item, some responses rejected the idea of lucky numbers, and attempted to express the chance element involved, but did not clearly express equality of chance: these were classified as transitional U-M_1 responses.

U-M_1 [Tattslotto]: It was just lucky and they aren't lucky numbers. [Year 3]

U-M_1 [Tattslotto]: No it is the luck of the draw. [Year 5]

At the M_1 level, responses expressed reasons for James' success other than the side of the bed, or claimed other numbers had the same chance for Claire.

M_1 [Bed]: I think it doesn't matter, to get good marks you should study. [Year 5]

M_1 [Bed]: It doesn't matter what side of bed you get out of, except if your bed is up against a wall. [Year 5]

M_1 [Tattslotto]: Every number has an equal chance of coming up. [Year 5]

M_1 [Tattslotto]: She still has the same chance as everyone else. [Year 5]

Responses at the R_1 level rejected James' and Claire's belief in luck, saying that luck was not the mechanism, but using psychological reasons for justifying their action.

R_1 [Bed]: I think that if James thinks it increases his marks then he will do better work and get better marks, the bed doesn't have anything to do with it. [Grade 5]

R_1 [Tattslotto]: If she won with it one week and she didn't take them again and they came up she would be angry so I would take them again. [Year 5]

Table 2

Percentage of Responses by Year to Items about Luck

SOLO Level	Q5 Lucky side of the bed Year 3	Q5 Lucky side of the bed Year 5	Q6 Tattslotto numbers Year 3	Q6 Tattslotto numbers Year 5
NR/Irrelevant	10	4	10	5
IK - Yes lucky (or "Yes")	17	10	16	12
U_1 - Not lucky (or "No")	69	75	67	62
U-M_1 - Luck of draw	-	-	7	20
M_1 - Same chance	5	10	<1	2
R_1 - Psychology	1	2	-	<1
N	864	703	864	703

Table 2 contains the results for Years 3 and 5 for these questions, where it can be seen that while students in these years do not offer structurally complex responses, over 80% of Year 5 and over 70% of Year 3 students reject beliefs associated with luck.

Understanding of Chance Measurement (Items Q7, Q8 and Q9)

Responses to the items about chance measurement (see Figure 1, Q7, Q8 and Q9) were categorised as expressing basic uncertainty, qualifying chance, and quantifying chance, as described by Watson et al. (1997). At the ikonic level, personal experiences or intuitions were used to make decisions about events with uncertain outcomes, often with idiosyncratic beliefs.

IK [Die]: (1), One is an unlucky number for some people in games. [Year 3]

IK [Names]: (b), Because boys are better than girls. [Year 3]

IK [Marbles]: (=), Because I like blue better. [Year 3]

At the U_1 level there was recognition of basic uncertainty in the outcome, often expressed as "anything can happen."

U_1 [Die]: (=), You never know what number you will throw. [Year 5]

Ul [Names]: (=), You don't know if it will be a boy or a girl. [Year 3]

U_1 [Marbles]: (=), Because you aren't looking and it could be either. [Year 5]

At the M_1 level chance was qualified in some way, such as indicating which outcome is more likely, or if the chances are the same for two outcomes. Some responses expressed conflicting ideas in qualifying chance.

M_1 [Die]: (=), It is just as easy to throw either number. [Year 5]

M_1 [Names]: (g), Because there's more girls in the hat. [Year 3]

M_1 [Names]: (=), It is 50-50 because there is only a difference of 3. [Year 5]

M_1 [Marbles]: (B), Box B has more blue marbles. [Year 3]

M_1 [Marbles]: (B), Because there are more marbles. [Year 5]

At the R_1 level, chance was measured numerically for simple settings. This level did not clearly account for the more complex setting involving two boxes in Q9.

R_1 [Die]: (=), It is a one in six chance for both of them. [Year 5]

R_1 [Marbles]: (A), Because in box A there are only two more red than blue but in box B there are 20 more. [Year 5]

R_1 [Marbles]: (=), Because there is more red in each box. [Year 3]

Functioning in the second cycle of the concrete symbolic mode was required to justify the correct response of (=) for Q9 with appropriate mathematical reasoning (note the inappropriate reasoning in responses above which chose (=)). At the U_2 level responses displayed a single idea consistent with the use of ratio, but without further justification or explicit reference to measurement.

U_2 [Marbles]: (=), You've got the same chance of getting a blue or a red, so any. [Year 5]

At the M_2 level, responses often compared numbers across boxes in an attempt to describe the relationship between the boxes.

M_2 [Marbles]: (=), If you pick A you would probably get a red and it's the same with box B except there's 10 x more. [Year 5]

At the R_2 level, responses needed to present a correct mathematical quantitative comparison, using ratios and/or percentages to measure chance within each box.

R_2 [Marbles]: (=), Because you have a 40% chance of getting a blue marble in each. [Grade 5]

Results for items Q7, Q8 and Q9 are shown in Table 3. Between 18% and 20% of Year 3 students and between 10% and 14% of Year 5 students gave responses to the multiple choice items but did not justify their responses with reasoning. These responses could not be allocated to SOLO levels. For Q7 the modal response for each year was associated with ikonic beliefs about dice. For Q8 however, in both groups over half of the students made sensible qualitative statements about the chances involved. For the more difficult Q9, only 11% of Year 3 and 21% of Year 5 students gave responses justifying the correct multiple choice selection which were in the second cycle of the concrete symbolic mode.

Table 3

Percentage of Responses by Year to Items about Chance Measurement

	Q7 Six-sided die		Q8 Names in hat		Q9 Marbles in boxes	
SOLO Level	Year 3	Year 5	Year 3	Year 5	Year 3	Year 5
NR	3	4	4	2	4	3
Incorrect choice	8	6	11	6	10	6
Correct choice only	10	8	7	4	10	7
IK	57	33	15	10	4	2
U_1	14	24	13	16	10	10
M_1	7	22	51	62	52	43
R_1	<1	4	-	-	7	6
U_2					3	12
M_2					1	2
R_2					-	<1
N	864	703	864	703	864	703

Understanding Data in a Table (Item Q10)

Q10 (see Figure 1) was intended to provide base-line information on data interpretation skills for the youngest students. Parts (a) and (b) were U_1 level questions for picking individual data values from a table. Part (c) was a straightforward M_1 question requiring the sum of numbers in two cells. Although an explanation was not sought for Part (d) and hence guessing was possible, it is likely that R_1 reasoning is required to compare girls and boys across each category. Part (e) was a more complex M_1 question which also asked for an explanation. The results for each part of the question are given in Table 4, with explanations for Part (e) in Table 5. Virtually all students performed at the M_1 level or higher on Q10, but considerably fewer students demonstrated correct calculations when totalling in Part (e). Many students gave a correct answer of "girls" by adding or counting but did not produce their totals. A few students in the "Add/count group" used a calculator. "Other reasons" included those choosing girls because they participated in every sport while boys did not.

Table 4

Percentage of Responses Correct to Item Q10 by Year

Correct	Year 3	Year 5
(a) 15	92	94
(b) 0	95	99
(c) 30	89	96
(d) Tennis	55	80
(e) Girls	71	81
N	864	703

Table 5

Percentage of Responses by Reason for Item Q10e by Year

Reason	Year 3	Year 5
60 and 58	15	10
60 or 58	9	3
Girls, Add/Count	35	54
Other, Add/Count	9	8
Other reasons	31	26
N	864	703

Discussion

The results of these analyses point to several useful findings for those who would describe the capabilities of Year 3 and 5 students in relation to chance and data. It is evident that nearly all students by Year 3 can successfully read straightforward information from tables. Most have a basic foundation for building a more sophisticated understanding of average and sample, and have understood that belief in luck is contrary to the operation of chance events. It also appears that chance situations involving dice present more difficulty for interpretation than chance situations involving simple sampling, such as from a hat. This may be related to children's experiences playing games with dice, for example when they need a certain number to start. It was also found that performance improved between Years 3 and 5 for each of the items used (noting the ceiling effect for Q10). It would appear that students in Years 3 and 5 are building the foundations of understanding to handle the learning experiences associated with the objective expressed in *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991).

> A sound grasp of concepts in the areas of chance, data handling and statistical inference is critical for the levels of numeracy appropriate for informed participation in society today. (p. 163)

The questions in this study were designed to be relevant to older students as well as Years 3 and 5, in order to trace the development across all of the years of schooling. Hence many trivial questions were not included. Asking other questions of a more basic nature, relating to simple outcomes from chance events, sampling and interpretations of graphs, would have provided different information which might be used for benchmarking. As part of the larger study which collected these data, interviews were conducted with over 100 students using such questions, and this information will further inform the understanding of children's development in Years 3 and 5. As well it will be possible to describe longitudinal development over time (e.g., Watson & Moritz, 1998). As the curriculum becomes more firmly established, future research involving teaching experiments will reveal how much more students can achieve.

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"This is a funny game - you can't say who's going to win!": Three case studies of children's probabilistic thinking

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Three case studies of children are used to illustrate the variety of strategies employed by children when asked to make probability judgments in several different game contexts. The children's responses ranged from idiosyncratic and intuitive reactions to the deliberate application of proportional reasoning. It was found that certain combinations of variables in the task designs stimulated different mathematical thinking.

Introduction

In an exploratory study of children's probabilistic thinking, task based interviews with 74 children, aged from 5 to 12 years, were conducted. This paper reports the information gathered about three of the children; Jessica (5 years and 11 months), Jason, (9 years and 11 months), and Jack (11 years and 8 months). The children were selected from the 74 simply because their interview responses covered a range of the types of thinking observed in the children, and were also representative of the type of responses in their age groups. These children were considered by their teachers to be of average mathematical ability.

The key research questions for the exploratory study were:

1. What strategies do children utilise for making judgments in different types of probability tasks?
2. What is the relationship between these strategies and the type of probability task?
3. Can the children's responses be classified into the expected developmental stages of non-probabilistic thinking, estimation of probability and quantification of probability?

In order to gain information which might provide some answers to these questions a set of interview tasks, in the form of games, were designed around the following variables: the type of random generator to used (*numerical* - objects in a jar or *spatial* - spinners); the structure of the sample space involved (such as the number of colours used and the relative amounts of each; the nature of the comparison being requested (such as within the one sample space or between sample spaces); and the type of responses expected from the child (usually a choice from two or more items and a reason for the choice). The identification of these variables, and the design of the tasks was influenced by the work of a number of people, including; Fischbein and his colleagues (1970), Hoemann & Ross (1971), J.Truran (1994), and K.Truran (1995). A range of specific concepts associated with probability were covered by these tasks. These concepts are listed in Table 1, together with a summary of the structure of each interview task.

The Interview Tasks

Task 1: Bears in a Box : Four small coloured bears (3 of one colour, 1 of another colour) are placed in a box. The child is asked to say which colour is most likely to be drawn out, and why the choice has been made. The child draws out a bear and then replaces it. This is repeated 5 times, with a display kept of each outcome.

Task 2: Non-replacement : This is similar to the first task except the bear is not replaced after each draw.

Task 3: Racing Cars : Four spinners of differing construction are used to play a game where coloured discs are moved along a race track. The child is asked to make various choices in regards to the most likely winner using a particular spinner, or the best spinner to use to get a certain result. Reasons are sought for each choice.

Task 4: Transfer : The child is asked to place coloured bears in the box to replicate the colour ratios of the spinners from the Racing Car game and to explain how they decided on the number of bears of each colour.

Task 5: Proportions : Two jars of various mixtures of red and yellow bears are displayed to the child, who is asked to choose the jar that will give the better chance of drawing out a red. The coloured bears are lined up outside the jars to facilitate the child's choice and explanation.

Table 1: Summary of Analysis of Interview Tasks

Interview Task	**Type of Random Generator**	**Structure of Sample Space**	**Nature of Comparison**	**Type of Response**	**Specific Probability Concepts**
Bears in a Box	numerical	2 colours 3:1	Intra-sample space	Choice of colour Reason	Sample space - constant Randomness More likely
Non-replacement	numerical	2 colours Initially 3:1 then changing	Intra-sample space	Choice of colour Reason	Sample space - changing (Simple conditional probability) Randomness More likely Equally likely
Racing Cars	spatial	4 or 3 colours A 1:1:1:1 B 4:2:1:1 C 3:1:2:2 D 3:1:4:0	Intra & Inter-sample space Same type of RG	Choice of colour Choice of sample space Reason	Sample space & Randomness More likely/Most likely Less likely/Least likely Equally likely Impossible/certain Spatial proportions
Transfer	numerical & spatial	4 colours A 1:1:1:1 B 4:2:1:1	Inter-sample space Different types of RG	Choice of numbers of items Reason	Sample space & Randomness More likely/Most likely Less likely/Least likely Equally likely Impossible/certain Proportions, spatial & numerical Quantification of chance
Proportions	numerical	2 colours 1:4 & 3:4 2:4 & 4:8 1:4 & 3:6 1:3 & 2:8	Inter-sample space Same types of RG	Choice of sample space Reason	Sample space & Randomness More likely Proportions, numerical Quantification of chance

Responses to the Tasks and Discussion

Task 1: Bears in a Box

Jessica: (5.11yrs) Jessica placed 3 Red(R) and 1 Yellow (Y) bears in the box, but chose Y as the more likely outcome for the first draw, saying "*'cause I like it*". Before the third draw (after drawing a Y then a R) Jessica chose R and gave the reason, "*I like it very much and that's my second, um third best colour*". After drawing a Y and a R, her third response was to choose Y again, saying "I like yellow". The final two draws resulted in a R and a Y. Jessica then predicted that another six draws would produce different results. She explained it would be different by saying, "*There's three red and three yellow, so that's six - red, red, yellow, yellow, red, yellow*". Jessica's reply to the question about why the box shaken before each draw was, "*To mix them up*".

Jason: (9.11yrs) Jason chose to put 3 green (G) and 1 red (R) into the box. When asked to say which colour he thought he was more likely to draw out, he said, "*Green,* *'cause there's more of green"*. His first draw resulted in R, but his second draw produced a G. When asked to again predict, he responded, *"Green, 'cause there's more of it"*. Following two more draws of R then G, the third question elicited the same response. The final two draws were R then G.

Jason said that he thought another six draws would produce a different result, *"'cause you might pick different ones - maybe 4 Green and 2 Red"*. Six more draws gave the sequence RRRGGR, to which he exclaimed, *"Wrong way around!"* Jason's answer in regards to the box shaking was, *"Stirring them around so you get different ones"*

Jack: (11.8yrs) Jack's selection was 3 Blue and 1 Red, and he correctly specified Blue as the more likely outcome of each draw, with comments of: *"There's three blue and only one red"*, *"Still blue. It's just chance. There's more chance to get one of the blue than the one red"*', and *"Blue - same reason"*. The results of his draws were RRBBRB. He decided that six more draws would produce different results and said, *"Mainly just get blues"*. The results were RBRBBB. When asked why the box was shaken Jack responded, *"So that you wouldn't know were they were in the same position and so keep pulling the same one out"*.

* * *

Jessica's responses indicate non-probabilistic reasoning. She does not appear to realise the connection between the structure of the sample space and the likelihood of various outcomes. Although there is some acknowledgment of the concept of randomness, the understanding appears weak. Although Jessica said the next six draws would be different she seemed to believe there would still be three Red and three Yellow drawn.

Jason's responses were classified as being at the *Estimation* level because they clearly utilised knowledge of the structure of the sample space, were not influenced by the outcomes of each draw, but did not involve any quantification. He showed an understanding of the concept or *more likely* and also of the idea of *randomness*. The data sequence generated by the draws did not influence his predictions which suggests he has a sense of the independence of each draw.

Jack's responses suggest a willingness to use numbers to describe likelihood. He exhibited a strong understanding of the way in which knowledge of the structure of the sample space can be used to determine likelihood, yet also understands that the outcome of each draw remains essentially unpredictable.

* * *

Task 2: Non-replacement

Jessica: With three Red and one Yellow in the box to start with, Jessica predicted Y each time saying, *"Because it's a good colour"*' and *"Sometimes it's on flowers"*. The equally likely situation did not arise, even though the game was repeated twice. However, it was apparent that Jessica was able to keep mental track of the contents of the box after each draw.

Jason : Jason again placed three Green and one Red bears into the box, and again stated that G was more likely to be drawn, *"'cause there's more of them."*. A Gr was drawn, but not replaced. Jason's prediction was G for the next draw, *"'cause there's still more of them"*. A G was once again drawn. When asked which colour he thought more likely to drawn next he replied, *" It's a lucky guess."* ("Why?", asked the interviewer). *"They're the same in there. I might get the Red"*.

Jack : With 3 Blue and 1 Red in the box, Jack chose B as the more likely outcome, and a B was drawn. Before the next draw Jack again chose B, saying *"There's at least one more chance to get blue than red"*. A R was drawn next. Jack's reply to the final question was, *"It's even, there's one of each"*.

* * *

Jessica paid little attention to the structure of the sample space and gave no indication of an awareness of the differing likelihoods of outcomes.

Jason had no difficulty in keeping mental track of the contents of the box, that is, the changing sample space, and so his responses implied an understanding of very basic conditional probability. His last response indicates recognition of equal likelihood.

Jack acknowledged the changes in the sample space and the impact on the likelihood of the possible outcomes. He also was able to recognise equal likelihood.

Task 3: Racing Cars

Jessica: After playing a game with the Spinner A (all colours equally likely) to establish the context of the task, Jessica was shown Spinner C and asked to say what colour car would have the best chance of winning the race, playing with that spinner. Jessica responded incorrectly by saying, "*Green, because you won the last race with Green. This is a funny game - you can't say who's going to win!*". She lost the game.

Next, Jessica was told she must be the Red car, but could choose the spinner she thought would give her the best chance of winning the race. She was asked to select the spinner from the whole set of four spinners. She correctly chose Spinner D, saying, "*It's got the highest bit there* (pointing to the red half of the spinner)". She won the game played with this spinner.

Jessica was then asked to select, from the set of four, the spinner that should be used for a *fair* game; one in which each of the four colour cars would have the *same chance* of winning. She gave the reason, "*They're all got the same shape*". (No more games were actually played).

Next, Jessica was asked if there was a spinner that would make it *impossible* for the Green car to win the race. She chose correctly, with the explanation, "*I think it would be hardest for Green - it's got none!*".

The next question required Jessica to identify the spinner that would give the Yellow car the *best chance* of winning the race. This was correctly chosen, with the statement, "*It's got the biggest yellow*". Jessica was quite certain that yellow would win and explained, "*Yes, because if I spin it will go there, there, there on yellow all the time (pointing to several locations on the yellow sector*".

Finally, Jessica was asked to identify the spinner that would make Red the *least likely* to win, that is, the best spinner to make Red lose the race. Jessica chose correctly, saying "*Because it's got the tiniest red*". She was also certain that Red would lose - simply saying, "*Yep!*".

Jason: Jason correctly chose the Blue car after looking at Spinner C, giving the explanation, "*'cause there's more space of Blue than the others - you might spin it. It's got more chance of landing there*". Jason then played the game as the Blue car and won the race.

Jason correctly chose Spinner B for the Red car to win with, saying "*It's got a full half. That one's* (Spinner D) *not quite half*". Jason played the game as the Red car and won again.

Jason correctly selected Spinner A as the *equally likely* spinner saying, "*'cause it's got the same amount of each one*".

Jason explained his choice of Spinner D for impossible-for-Green by saying, "*There's none on it - there's no green there.*"

He correctly chose Spinner D as best-for-Yellow because, "*It's got more - more yellow on there*". When asked if was certain that Yellow would win he responded, "*No - 'cause the other cars 've got a chance as well.*"

Jason picked Spinner C for Red-to-lose and explained, "*It's got the least red on it*". When asked if it would lose for certain, he said, "*No - it's still got a chance of winning.*"

Jack: Jack's reason for choosing the Blue car for Spinner C was "*It's got a large percentage of the circle - it's got more of a space where the arrow can land*".

Jack explained his choice of spinner for the Red car by saying, "*It's half Red and there's three others (colours) to make the other half*".

The equal chance spinner was correctly identified and Jack said, *"It's all split up into quarters - everyone's got the same percentage"*.

The choice of spinner for impossible-for-Green was justified by, *"It's got no Green on it"*.

Jack correctly chose the spinner most likely to produce a Yellow win, saying, *"It's got half of Yellow and there's two (colours) to make up the other half"*. When questioned about the certainty of the result of a game he said, *"It's not 100%. It has to be definite - it's got a 50/50 chance"*.

Jack explained the correct choice of the Red-to-lose spinner by saying, *"It's the lowest percentage of Red"*. When asked about the certainty of Red losing he replied, *"There is red on the card, if there was no red it would lose"*.

Jessica's first response relied on the result of the previous game rather than the structure of the sample space, and therefore indicated non-probabilistic reasoning. However, the rest of her decisions were based on the size of the colour sectors on the spinners, showing that she was able to make successful comparisons between sample spaces of this type. Her responses are categorised as *Estimation* strategies. Although Jessica noticed the unpredictability of the outcome of a game earlier, her belief in the certainty that her predicted outcomes would occur indicate an incomplete understanding of the notion of randomness.

Jason's series of responses indicate that he felt quite comfortable in making comparisons among several sample spaces, not just focusing on the outcomes of a single sample space. He demonstrated an understanding of a range of probability concepts, including *most likely, least likely, equal likelihood, impossibility* and *certainty*. Most of his responses were at the *Estimation* level, which was sufficient to clearly explain his thinking in terms of *spatial proportions*. However, in one response, he did successfully use fractions to make comparisons between two sample spaces, and therefore reasoned, in this particular situation, at the *Quantification* level.

Jack, like Jason, made the required comparisons both within and between sample spaces with ease and demonstrated an understanding of a range of probability concepts. However, Jack also showed confidence in using fractions as part of his reasoning, which placed most of his answers in the *Quantification* category. There is little doubt that Jack has a sound understanding of the concepts of certainty and impossibility in this context.

Task 4: Transfer

Jessica: In this task, Jessica was presented with the scenario that we wanted to play the Racing Car Game but had lost all the spinners. Could we put some Bears into the box, so that the colour drawn out would tell us which colour car to move forward? Jessica appeared to understand the nature of the task, and when asked to create a fair game she responded, *"All of them. I'd put 3 blue, 4 yellow, 2 red, 5 green. That's if you were Green. But 5 of each for fair"*. Her answer to the impossible-for-Green question was, *"Some of the other colour and none or one of Green"*.

Next, Jessica was shown Spinner B and asked to work out how many of each colour bear to put into the box to make the game 'work the same way' as the spinner would. Jessica lined up four bears along the edge of one of the sectors on the spinner - a radius of the circle. She placed four of each colour bear into the box and explained, *"I counted it up from down to up"*. When asked if the box would be likely to produce the same winner as the spinner Jessica said, *"I think I'll put some more extra reds in - so Red should win, but I don't care about the others - but Yellow looks like it should have a bit more."* The final contents of the box were 6 Red, 4 Yellow, 4 Blue, 4 Green.

Jason: Jason understood the context and did not hesitate in placing one bear of each colour (red, blue, green, yellow) into the box when asked to 'set up' a *fair* game. When asked if he could put some more bears into the box but still keep it fair, he said, *"Two - as long as they're the same."*

Jason was then asked what he would put into the box to play a game in which it was *impossible* for Green to win the race. He simply stated, "*No green*".

When asked to replicate Spinner B, Jason placed 4 Red, 2 Yellow, 1 Blue and 1 Green into the box, which correctly matched the ratio on which the spinner was designed. When asked to explain his thinking Jason said, "*Blue and green had the least so I put in one, and yellow had the second most so I put in two - and that one's* (Red) *got four*". "Why?" "*Four of those* (Blue sector) *can fit in there*".

Jack: Jack's initial response when asked to place bears in the box for a fair game was, "*Seven of each colour*". When asked if there was another amount that could be used he replied, "*You can put one of each, but you'd have to keep putting them back in*". Jason had at first assumed that the bears would not be replaced after each draw and so had put in one bear for each race-track space so that each colour car would have the opportunity to reach the finish line.

For the *impossible*-for-Green spinner Jack simply said, "*Don't put any Green in*".

To replicate Spinner B, Jack began by saying, "*Five red*", then paused for a while to think. "*I'm going to have to chop some in half. 5 red make up a half,* $2\,^{1}/_{2}$ *for yellow,* $1\,^{1}/_{4}$ *for blue and green*". The interviewer asked if changing the number of red bears would make it possible to actually put the right number of bears into the box. "*Probably could*", Jack replied. When asked why he chose to use five he explained, "*Scaling down from 100%, make it 10 bears go in* (to represent the total circle), *so half is 5*".

Jessica's responses in this task suggest the beginning of understanding in some probability concepts. She had successfully identified the *fair* spinner in the previous task, and now, after a shaky start, was able to construct her own *equal likelihood* sample space using number. Jessica's comprehension of the meaning of impossibility in relation to a sample space was once again shown to be incomplete. She thought that having only one Green in the box would still guarantee the Green to lose the race. In the final question, Jessica at first attempted a simple measurement strategy, but when she realised that this had not been successful, she made an incomplete attempt to relate the size of some sectors to the number of bears of that colour. Although her focus appeared to be on the winner of the game, there was an incomplete attempt at ordering the likelihood of each race outcome.

Jason was able to once again illustrate the notion of equal likelihood, this time in a way that showed his ability to work with proportions. He also demonstrated that his understanding of impossibility was not limited to the spatial random generator (spinner), but could be modelled with discrete items (bears). Although, in response to the final question, Jason did not use the language of doubling, halving or fractions, it is clear that he was 'unitising'. That is, he was using the smallest sector in comparison to the other sectors, in order to quantify their relative sizes - he was building ratios. Jason had little difficulty in comparing the two different types of random generators.

Jack's responses to the equal likelihood questions demonstrated his understanding of basic conditional probability because he was able to think through the consequences of non-replacement after draws, and apply this understanding to design an appropriate random generator for the game. Jack used a part/whole approach to quantify the size of the sectors on Spinner B. He began with a number to represent the whole circle, then used the fractions he estimated by looking at the spinner to determine the number of bears required for each sector. That is, he began with 10 for the whole circle - he could *see* that the red sector covered *half* the circle so he *halved* 10 to determine there should be 5 red bears in the box, and so on.

Task 5: Proportions

The final task was designed to further promote numerical thinking and hence provide the opportunity for the quantification of chance. Unlike the spinners, the jars of bears in this task could represent a wide range of ratios, with a variety of denominators. It must be remembered that there is a 50% chance of choosing the correct jar, and it is the reasoning behind the choice that provides the most useful information about the child's thinking.

Jessica: Game 1: Jessica was asked to choose between jars containing the following ratios; 1:4 and 3:4. The second jar was correctly chosen and the reason given was; *"The second one looks easier. There's 3 red and 1 Yellow".*

Game 2: The jars contained 2:4 and 4:8. Jessica incorrectly chose one jar (the second one) saying, *"It's got the most of reds".*

Game 3: When shown the 1:4 and 3:6 jars, the correct jar was chosen, with the explanation, *"There's about 3 reds and if I get 3 reds I'll win".*

Game 4: The jars contained 1:3 and 2:8. Jessica incorrectly chose the second jar, saying *"It's got 2 reds and if I get that I win another game".*

Jason: Game 1: Jason correctly chose the second jar, saying, *"This one would have more chance of getting reds 'cause it's got 2 more than the other jar, and yellow's would be the same".*

Game 2: Jason responded correctly. *"Um... Yellow's more likely...For red...they're both the same because there's double the amount of red in that jar, and there's double the amount* (of yellow) *in that jar".*

Game 3: Jason incorrectly stated, *"They're both the same because that one's* (1st jar) *got 3 more yellow* (than red), *and that one's* (2nd jar) *got 3 more yellow than red too".*

Game 4: Jason correctly chose the first jar, but not for the most appropriate reason, "*'cause this one's* (1st jar) *only got 3 more* (yellow than red) *and this jar's* (2nd jar) *got 6 more* (yellow than red)".

Jack: Jack chose the correct jar/s in each game. His reasons are listed below.

Game 1: (1:4 and 3:4 jars). *"This jar* (2nd jar)*would give the best chance at picking out red 'cause there's 3 red and in the other jar there's only one red".*

Game 2: (2:4 and 4:8 jars). *"They've both got half the percent of red of the yellow".*

Game 3: (1:4 and 3:6 jars) "*'Cause that's* (2nd jar) *half* (red half of yellow), *and that's* (1st jar) *only quarter."*

Game 4: (1:3 and 2:8 jars). *"This* (1st jar) *is one third and this* (2nd jar) *is one quarter".*

Jessica chose the second jar in every game and there is little evidence that she was able to make any numerical comparison between the two 'whole' sample spaces. There was a suggestion of the use of visual estimation when she says "It looks easier" and "It's got the most of reds", a strategy that was successful for her in the *Racing Car Task*. However, her main strategy was to choose the jar with the greater number reds in it, regardless of the number of yellows. In other words, she only used the *favourable outcome* part of the sample space in her reasoning, so there is no perception of ratio evident.

Jason used several different strategies in his decision making. In the first game he simply compared the quantity of the favourable colour (red) in each jar. This was appropriate,because the number of yellow bears in each jar was the same. In the second game, Jason used doubling to explain the equivalence of the ratios contained in jars - not unlike the thinking he used to create *equally likely* ratios in the previous task. In the third game, Jason first compared the quantity of red and yellow within each jar to find the difference score for each jar, then compared these two scores. When he realised the differences were the same for both jars he concluded that each jar must present the same likelihood for red being drawn. This strategy is quite unlike any he had previously used. In the fourth game, it appears that Jason is again trying a subtractive strategy but incorrectly calculates one difference.

Jack once again applied his understanding of fractions to the comparison of sample spaces. He was able to state the ratio of red to yellow (in the form of a fraction) and then compare these in terms of the likelihood of drawing a red. His ability to use proportional thinking was highlighted by the recognition of equivalent fractions in Games 2 and 4.

Conclusion

The three children used a range of strategies in their decision making. Jessica's strategies were mostly non-probabilistic and her thinking idiosyncratic. She did, however, demonstrate some early understanding of some basic concepts that underlie probability. Jessica found it easier to make comparisons when the sample spaces were represented on the spinners. Jason's understanding of probability was more developed and he was able to use probability estimation strategies quite effectively, except in the final task. Jason's attempts to apply his emerging understanding of fractions and ratio indicate a readiness to learn new skills in this area. Jack's more sophisticated understanding of fractions and ratio equipped him to confidently find effective decision making strategies to suit each task, and enabled him to quantify probabilities. Both the older children illustrated the importance of developing doubling and halving skills, as these provided the foundation for several successful strategies. The final two tasks were clearly more challenging than the others, and accordingly, stimulated more complex mathematical thinking.

The differences in performance could be accounted for by developmental theory such as Piaget and Inhelder's (1951). However, the different levels of understanding can also be described in terms of the number and type of relationships the children were able to perceive between the variables in the tasks and their existing mathematical knowledge, such as used by Watson & Collis, 1994. None of the children had received any systematic instruction in probability; it not being a topic included in the curriculum. Therefore, the effectiveness of the strategies invented and applied by the children not only depended on their intuitive understanding of probability concepts and their abilities to work with fractions, ratio and proportion, but also on their ability to perceive the connections between the range mathematical elements involved in the tasks. The ways in which the task variables identified in this study (see Table 1) can be manipulated to stimulate various types of mathematical thinking in different age groups warrants further investigation.

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Beliefs about the Use of Calculators in an Upper Primary Mathematics Classroom: A Partial Application of the Theory of Planned Behaviour

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This study surveyed a sample of teachers ($N = 115$) with a questionnaire using the theory of planned behaviour (TPB). It examined primary school teacher intentions to allow students to use calculators in the classroom and the influences upon these intentions. Key salient behavioural, normative and control beliefs held by the teachers are analysed and evaluated according to their respective contribution to attitudes, perceptions of influence by the social environment, and perceived behavioural control.

Although calculators have been available since 1965 their progress into schools has not been an orderly transition and instead has attracted considerable controversy. In 1986, a UNESCO study lamented the lack of Australian research upon calculators (Blane & Willis, 1986), yet the research that had been completed locally and overseas reflected strong opposition to their use in primary schools and this early opposition has tended to exert a continuing influence (Howard, 1992). Research and debate over student use of calculators has ranged across issues such as: access and equity; the impact upon basic skills; and the use of calculators to enhance the teaching and learning contexts. Of interest is how have primary school teachers been influenced by this debate? What do they believe about student use of calculators and who has the greatest influence upon their beliefs? Research is needed in this area to provide information to assist in the understanding of teacher behavioural intentions and to assist teachers in their process of critical reflection of current practice. A recent study (White, 1996; White, 1997; White, in press) uncovered and categorised the range of beliefs held by NSW primary teachers towards the use of calculators by their students. However this study was unable to indicate which of these beliefs provided the basis for teacher intentions or behaviour. This current study attempts to do this by applying the constructs and instruments of the model titled the Theory of Planned Behaviour (TPB). It aims to sort the beliefs uncovered by the earlier study indicating their relative importance in the formation of teacher behavioural intentions.

Theoretical Issues

The Theory of Planned Behaviour (TPB; Ajzen, 1985, 1987, 1988) is an extension of the Theory of Reasoned Action (TRA; Fishbein and Ajzen,1975) and both have been very successful in predicting behaviour over a wide range of contexts and both come from an objectivist (positivist) stance. This stance has been the target of a great deal of criticism (see for example Lather, 1991) and it is beyond the scope of this paper, however some brief comment is provided. This study does not adopt a positivist perspective in its use of TPB as it is not interested in predicting behaviour. It uses TPB to sort the teacher beliefs and to indicate their relative importance in the formation of teacher behavioural intentions. Thus the use of TPB resonates with an interpretative paradigm. Not surprisingly there are differing opinions about the wisdom of mixing paradigms. Whereas Guba and Lincoln (1988) believe that paradigm distinctions are real and critically important to research, others such as Cook and Reichardt (1979) believe that they are artificial and irrelevant to method choice and others such as Entwistle (1984) emphasise the crucial importance of the perspective and intention of the researcher.

According to TRA the immediate determinant of behaviour is intention and two major factors determine a teacher's behavioural intention: a personal or attitudinal component; and a social or normative component. The attitudinal component measures the teacher's attitude towards performing the behaviour and "is simply a person's general feeling of favorableness or unfavorableness for that concept" (Ajzen and Fishbein, 1980, p. 54). The second or normative component of the theory deals with the influence of the social environment upon intention and behaviour. This is a measure of the teacher's perception of whether most important people support or don't support the performance of the behaviour. This implies that a teacher will usually intend to perform a behaviour that is positively evaluated and has the support of significant others. If TRA was to stop at this point we would have very little information to assist our understanding of behaviour. However, both components can be investigated further. Attitude towards a behaviour is determined by the product of the teacher's beliefs about performing the action and the evaluation of the outcomes of the action. Subjective norm is determined by the product of the teacher's beliefs of social expectations to perform the behaviour and the motivation to comply with these expectations. In the final analysis then, TRA attempts to explain a teacher's behaviour in terms of beliefs which represent the information (be it correct or incorrect) that the teacher has about their situation. Successful predictions of behaviour using TRA have been done of such diverse areas as voting behaviour (Fishbein, Ajzen & Hinkle, 1980), family planning (Fishbein, Jaccard, Davidson, Ajzen & Loken, 1980), consumer behaviour (Fishbein & Ajzen, 1980), weight reduction (Sejwacz, Fishbein & Ajzen, 1980), changing alcoholic behaviour (Fishbein, Ajzen & McArdle, 1980), smoking behaviour (Fishbein, 1982), infant feeding (Manstead, Proffitt & Smart, 1983), seatbelt use (Budd, North, & Spencer, 1984), and continuing education (Prior, 1990)

TPB expanded TRA in order to include the construct of perceived behavioural control (PBC). Ajzen (1988) proposed that it would be important as a determinant of intention when the individual had previous knowledge or experience of the behaviour in question. In this study it is probable that teachers would have had knowledge or experience of the behaviour included in the questionnaire. Other researchers have had similiar views such as Budd, North, and Spencer (1984) and Wittenbraker, Gibbs, and Kahle (1983) who included past behaviour as a determinant of behavioural intention when using TRA. Fewer studies have been published using TPB, however Schifter and Ajzen (1985) studied perceived control and weight loss and Ajzen and Madden (1986) used the theory to predict undergraduates' course performance. Parker, Manstead, Stadling, Reason and Baxter (1992) made an assessment of the ability of TPB to account for the intentions of driver's to commit driving violations and concluded that "the addition of perceived behavioral control led to significant increments in the amount of explained variance of intentions, thereby supporting the theory" (p. 94). For the purposes of this paper, only the salient beliefs used in regard to the constructs of attitude, subjective norm and perceived behavioural control will be examined and reported.

Method

TPB requires behaviour to be carefully defined. This was done by means of the scenario below which was taken from the earlier study (White 1996; White, 1997; White, in press) as it produced a high level of agreement among teachers.

> *Scenario 1:* You are teaching mathematics to your class at the usual time and place. You want the class to complete an exercise. You consider a range of options before insisting that the students use calculators. (In this study a calculator can include any of the range of available handheld four operational models).

A questionnaire was constructed according to TPB consisted of 28 semantic differential scales and was applied to a sample chosen randomly from across the population of primary teachers who worked at a NSW Department of School Education primary school and who taught mathematics to years 5 and/or 6. Two letters of the alphabet were randomly selected and schools whose name began with those letters were eligible for selection providing they met certain conditions. The teachers that were approached, belonged to schools:

(i) that had an enrollment above 100 as listed in the 1993 Directory of Government Schools in New South Wales (DSE, 1993). The purpose of this condition was to eliminate the very small schools. It was felt that these schools would have composite classes with too broad a range of abilities in the one class. That is, the class would have three or more year levels within the one class (years 4-6). It is argued that the teaching strategies employed by the teachers at these schools are likely to be different from those employed in larger schools. A question was included to monitor the levels within the classes of the teachers in the study. A disadvantage, however of this condition, was the reduction in the number of country schools in the sample.

(ii) that were not designated as a 'special' (SSP) school. Again it is argued that the teaching strategies employed by the teachers are different from those employed in larger schools.

A total of 119 teachers returned the questionnaire and the data was entered into a spreadsheet and the software program Statview 4.0 was used to analyse the data. A decision was made not to include four returns because of the amount of missing data.

The questionnaire asked teachers to imagine behaving in the manner described by the scenario and to rate their response on each of the 28 scales. There was one scale for the direct measure of intention to perform the behaviour (endpoints likely/unlikely) and three taking a direct measure of attitude (endpoints good/bad, beneficial/harmful, wise/foolish) which were later combined into a single index (Direct attitude index - DAI). A belief-based attitudinal index (BBAI) was also calculated. The behavioural beliefs were selected from across the continuum developed in the earlier study and are listed in Table 1.

Table 1

Beliefs about Allowing Students to Use Calculators in the Classroom

My decision to allow my students to use calculators in the classroom:

1. would result in familiarity with their use
2. would promote laziness and dependence
3. would result in them working faster and saving time
4. would result in students just accepting answers and not thinking
5. would help them simplify complex problems and assist their problem solving
6. would make it easier for them to cheat

The strength of the belief was measured on a bipolar 7 point scale (endpoints likely/unlikely) with a score of 3 signifying a strong positive belief strength and -3 signifying the opposite. The evaluation outcome was also measured on a similar scale (endpoints good/bad). The BBAI was constructed using the summed products of the six beliefs and the resultant scores ranged from 54 which indicated a very positive attitude towards the behaviour in question to a score of -54 which indicated the opposite (refer to Table 3).

There was one scale to measure teacher perceptions of the overall influence of important others and a corresponding scale to measure teacher willingness to comply with this influence. The product of these two scales produced one direct measure of subjective norm (SND). A belief-based index of subjective norm (SNB) was similarly obtained by summing the products of each of four normative beliefs about salient referents (Principal/supervisor, Parents, Students, Other Teachers) with their corresponding motivations to comply. To complete the TPB model there were two scales (endpoints of full-control/no-control, easy/difficult) to measure perceived behavioural control and these were combined to give one index (PBC). Finally there were four questions which gathered information on the teacher's sex, teaching experience in years, approximate age, and year level now taught.

Results

The demographic results of the sample if summarised using modal characteristics indicated that the teacher was likely to be female; aged between 36 and 45; with 11 to 15 years teaching experience; and with a class of only year 6 students.

Table 2

Constructs of the Theory of Planned Behaviour: Mean Values and Standard Deviations

	Mean	Standard deviation
Attitude toward behaviour		
Direct. DAI	3.95	3.58
Belief product. BBAI	14.04	11.46
Subjective norm.		
Direct. SND	0.61	2.24
Belief product. SNB	2.23	7.44
Control belief. PBC	4.29	1.30
Behavioural intention.	1.06	1.78

Note: The table contains both the direct measures and the summed product measures of attitude and subjective norm. For attitudes the direct measure range is -9 to 9 and the product measure is -54 to 54. The subjective norm direct measure range is -3 to 3 and the product measure range is -36 to 36. The behavioural intention range is -3 to 3. The control belief range is -6 to 6.

The descriptive statistics of the constructs ofTPB listed in Table 2 show that the majority of teachers' intended to allow students to use calculators within the clasroom. These intentions were influenced by: the teacher attitudes towards the behaviour which were positive on both direct and belief-based measures; the subjective norm measures which indicated that overall, the teachers perceived encouragement by their social environment; and the perceived behavioural control index which suggested that teachers felt they had a high degree of control over the decision and that it would be fairly easy to implement.

While the direct measures such as DAI are sufficient for prediction purposes they do little to assist the understanding of the intention or behaviour. The belief-based product measures are more useful such as those listed in Table 3. BBAI was formed by summing six products of the belief with the corresponding outcome evaluation. Responses to the first belief were very strong with approximately 99% of teachers indicating that it was likely to happen and 1% who disagreed. Evaluations were also very strong with 99% of teachers rating it as good. The corresponding paired products produced a strong positive attitudinal measure for approximately 98% of teachers and a negative result for 2%.

Responses to the belief that using calculators in class would promote laziness and dependence saw 71% of teachers regarding this as unlikely, 14% undecided and only 15% regarding it as likely. Evaluations produced 30% of teachers rating it as bad , 30% undecided and approximately 39% rating it favourably. The paired products produced a mean that was slightly negative the result of pairs being 30% positive, 33% negative and a zero for 37% of teachers. .

Responses to the third belief indicated that approximately 85% of teachers felt that it was likely with 5% undecided and 9% rating it as unlikely. Evaluations produced 75% of teachers rating it as good with 10% undecided and 15% rating it as bad. The paired products produced approximately 75% of scores that were positive, 10% negative and a zero result for 15%.

Responses to the fourth belief indicated that 27% of teachers felt it likely that using calculators in class would result in students just accepting answers and not thinking,with 10% undecided, and 63% rated it as unlikely. Evaluations produced 17% of teachers rating it as good , 24% undecided and approximately 59% rating it as bad. The paired products produced a slightly positive mean from 42% that were positive, 30% negative and a zero for 28%.

Table 3

The Means, Standard Deviations, and Frequency Distribution of the product of Behavioural Beliefs and corresponding Outcome Evaluations (n=115).

Class	1 familiar with use	2 laziness dependance	3 faster work	4 not thinking	5 problem solving	6 easy cheat
9 to 7	42	5	18	3	26	4
6 to 4	65	11	49	24	63	8
3 to 1	6	18	19	21	17	17
0	0	43	17	32	7	56
-1 to -3	1	9	7	18	2	12
-4 to -6	1	26	3	14	0	16
-7 to -9	0	3	2	3	0	2
Mean	6.1	-0.4	3.3	0.3	4.8	0.1
St. Dev.	2.7	3.9	3.8	3.6	2.9	3.3

Responses to the fifth belief indicated 93% of teachers felt that it was likely that using calculators in class would help students simplify complex problems and assist their problem solving with 4% undecided and 3% rating it as unlikely. Evaluations produced 94% with a rating of good , 4% undecided and a 2% bad rating. The attitudinal product pairs were 92% positive, 2% negative and a zero for 6%.

Responses to the final belief indicated 22% of teachers felt it was likely calculators would make it easier for students to cheat, with 57% rating it as unlikely, and 22% undecided. Evaluations produced a 27% good rating, a 31% bad rating and 42% undecided. The products produced a slightly positive mean from pairs that were 25% positive, 26% negative and a zero result for 49%.

Table 4

The Mean, Standard Deviation and Frequency Distribution of the product of Normative Beliefs and the corresponding Motivations to Comply (n=115).

Class	Principal	Parents	Students	Other teachers
9 - 7	3	2	1	1
6 - 5	11	18	20	15
3 - 1	24	24	23	20
0	53	53	50	62
-1 - -3	17	9	13	10
-4 - -6	6	8	5	6
-7 - -9	1	1	3	1
Mean	0.7	0.5	0.6	0.4
St. Dev.	2.8	2.7	2.8	2.4

The normative beliefs derive from two sources. The overall SND involved the normative belief concerning Important Others which showed a 41% perception of approval, a 19% perception of opposition and a 40% zero result. The motivation to comply had with teachers indicating 47% wanting to comply, 29% who did not and 24% who gave a zero. The products produced a weak positive mean from pairs which were 30% positive, 20% negative and a large 50% were zero.

An examination of the belief-based measure SNB consisting of the summed products of four salient referents listed in Table 4, produced similar weak results. The four products had large zero scores and the means were very slight. An examination of the underlying beliefs such as the belief concerning support by the Principal/supervisor produced 40% of teachers who perceived support, 14% who perceived opposition, and 36% who recorded neither. The results for the Parents indicated approximately 28% of teachers who perceived support, 34% who perceived opposition, and a 38% zero result. Influence by the Students produced 54% of teachers who perceived support; 13% who perceived opposition; and 33% who recorded neither. And finally the results for the Other Teachers indicated approximately 44% of teachers who perceived support, 12% who perceived opposition, and 42% who recorded neither. Teachers also indicated that they were more inclined to comply firstly with the wishes of their Principal/Supervisor then parents, the other teachers and finally the students.

The results for PBC indicated that 98% perceived they had some control over allowing students to use calculators, with 50% indicating very strong control. The underlying beliefs indicated that the majority of teachers perceived almost full control over the behaviour and that it was quite easy to perform. The responses to the control belief were stronger in comparison to the belief about the ease of performance which had a far greater range.

Discusssion

In this study, the constructs of TPB were applied to the salient beliefs and referents identified by an earlier study. Teachers indicated a positive intention towards student use of calculators. According to TPB this was determined by attitude, subjective norm and perceived behavioural control. Both measures of attitude were positive towards the behaviour. An examination of the underlying beliefs revealed that the positive beliefs had the greatest impact upon attitude formation. Thus attitude was strongly influenced by student gains in familarity of use, speed of computational facility and the assistance gained in problem solving of complex problems. The beliefs involving negative outcomes resulted in greater zero scores and approximately even distributions of negative and positive evaluations, suggesting that some teachers thought that the outcomes would entail more advantages than disadvantages. Thus beliefs about calculators causing laziness, dependency, blind acceptance of answers, lack of encouragement of thinking skills and cheating did not dampen the attitude of the majority of teachers towards calculator use.

The influence of the social environment upon intention was positive and the salient normative beliefs indicated perceptions of strong support from their students and other teachers, moderate support from their Principal/supervisor and opposition from the parents. Teachers indicated that they were more inclined to comply with the wishes of their Principal/supervisor and parents than either the other teachers or the students.

The influence of perceived behavioural control upon intention was positive and indicated that the a large majority of teachers perceived that they had control over performing their intentions.

The importance of the constructs of TPB is best illustrated by an example. The normative belief about parental influence indicated that more teachers perceived a negative pressure than a positive one. As most teachers also indicated a wish to comply with the wishes of the parents, simplistically we would expect a negative contribution to the influence of the social environment. This was not so and the slight positive mean illustrates the strength of the model which lies in the product pairing of corresponding results. By keeping the measures paired, we are in a better position to correctly interpret the teacher's response. For example, a teacher who perceives a strong opposition by parents towards calculator use (score -3) but is strongly motivated against complying with this pressure (score -3), is likely to have a positive intention towards the behaviour (score -3x-3=9). This example cautions against making too great a use of either component of the product and supports the need for a model to deal with the complexity of social pressures and their associated motivations to comply.

Conclusion

The constructs of the TPB model were used to analyse a number of behavioural, normative and control beliefs uncovered by an earlier study. The results of the current study indicated that the majority of upper primary teachers surveyed intended to have their students use calculators in the classroom. An examination of these intentions revealed a positive attitude towards this intention, a perception of approval from the social environment, and a perception of personal control over the implementation of the intention. A further examination concentrating upon the salient beliefs allowed them to be sorted as to their contribution to attitude, perceived control or subjective norm. According to TPB these beliefs underpin teacher behavioural intention and ultimately teacher behaviour. Thus this examination of teacher beliefs, outcomes, evaluations, motivations to comply, attitudes, perceptions of social influence and behavioural control provides a useful input into the process of understanding the teacher intention of allowing student calculator use in the classroom. From the accumulation of data from such investigations, teachers and others will be in a better position to understand, to reflect upon and to possibly change the way their world operates.

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Metacognition within mathematics:
A new and practical multi-method approach

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The importance of metacognition for student learning has been widely acknowledged (Biggs, 1987, Birenbaum, 1996, Brown and De Loache, 1983, Pintrich, and De Groot, 1990, Schoenfeld, 1987, Wilson and Wing Jan, in press). But the practicalities associated with teaching for metacognition and monitoring metacognition are not clear. This paper is concerned about the assessability of metacognition within mathematics. It is asserted that unless metacognition can be assessed then it will exist as a theoretically sound construct but never be considered a viable part of the mathematics curriculum.

Introduction

This paper reports on the results of a PhD pilot study that has been conducted with upper primary students in the curriculum domain of mathematics. It reports on what students said they did when they solved different types of mathematics problems. The pilot study highlighted key aspects of students' metacognitive thinking and raises important methodological questions about the validity of assessing of students' thinking in mathematics. A new, practical multi-method interview has been developed as a result of the pilot study to meet the methodological challenge of researching metacognition. It responds to questions of legitimacy and meets the needs of classroom researchers.

Project Aims

The study aimed to identify strategies for assessing metacognition within mathematics. Three subsequent questions were used to focus the study:

1. Which assessment strategies are most effective?
2. What do these assessment strategies reveal about the nature of metacognition?
3. What is the relationship between metacognition and task type?

Rationale

For over two decades many curriculum documents have promoted the importance of metacognition for improving the educational outcomes of students (The Australian National Statement on Mathematics, Australian Educational Council, 1991, Baird and Northfield, Eds, 1992, Biggs, 1987, Ministry of Education, 1989, Stacey, 1990, Wittrock, Ed, 1986). But there is much confusion about the meaning of metacognition. The term 'metacognition' has not been clearly defined, little teacher reference material about metacognition has been written and even less assessment material is available to monitor metacognition. With little teacher support, an 'overcrowded curriculum' (Pigdon and Woolley, Eds, 1992) and a move to test the basics, it is argued that metacognition will never be considered achievable as a curricula goal.

Clarke (1988) argues that assessment should monitor the development of the attributes and capabilities that we value. This paper asserts that metacognition should be assessed within mathematics to increase the likelihood of it being taught. The development of 'teacher friendly' assessment materials of metacognition would improve the likelihood of metacognition being seen as important, viable and basic to student and curriculum development.

Metacognition and mathematics

Research in the field of metacognition has mostly focussed on gifted or disabled learners, often in the area of literacy. Much work has been conducted at tertiary levels or with adults. A number of researchers have asserted the centrality of metacognition for effective mathematical thinking, especially in relation to problem solving (Schoenfeld, 1990). In research studies investigating mathematics and metacognitive decision making, Goos (1993, 1994, 1995) claims an important link: 'Metacognitive processes are considered to be an important factor influencing problem solving performance.' (1993:1) Schoenfeld, 1987, Venezky and Bregar, (1990) agree that the ability to monitor one's own learning increases the effectiveness of problem solving.

Stacey claims that: 'Good problem solvers tend to show more meta-cognitive knowledge. ie an awareness of the processes of learning and mathematics.'(1990:6) Research in the field of mathematics and metacognition has reported that students having difficulties in mathematics do not use a range of cognitive or metacognitive strategies Cardelle-Elewar (1992) and Munro (1993). It is no longer considered appropriate for students to just practise and memorise a collection of unrelated facts. For students to become self-directed problem solvers, teachers need to teach students to monitor and regulate their own learning. These suggestions of an association between metacognition and able mathematics problem solvers, strengthens the case for investigating the assessment of metacognition and has important implications for teaching and assessment practices.

Defining metacognition and its functions

Despite a growing interest in the concept of metacognition over the last twenty years and attempts to define reflective and metacognitive thinking, a lack of clarity has existed (Brown, 1987 and Munro, 1993). The terms used in this paper are defined below after a brief review of the literature.

The literature associated with metacognition dates back to Dewey in 1933. Although he didn't use the word metacognition, he spoke of 'reflective self awareness',-the importance of active, persistent and careful consideration of beliefs and knowledge. Flavell (1976) was the first to use the term 'metacognition', which refers to the individual's awareness, consideration and control of his or her own cognitive processes and strategies. Since then a variety of meanings given to the term.

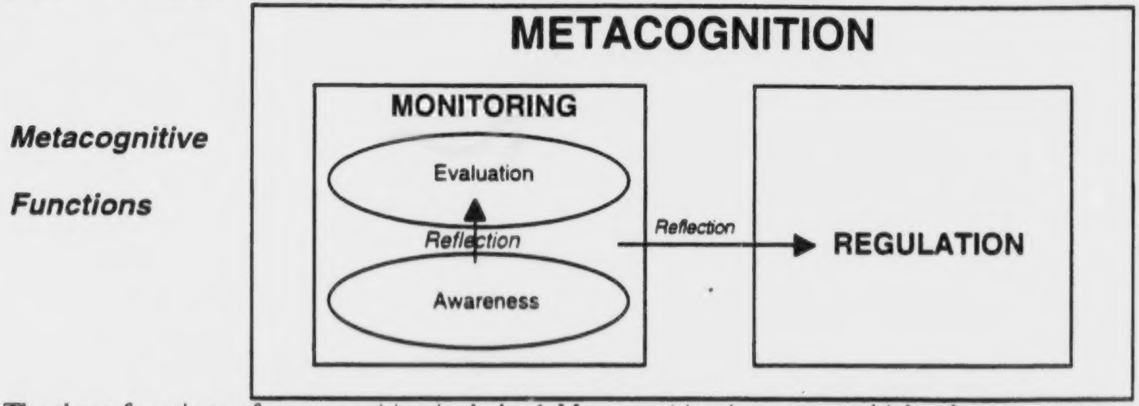
Perry (1989) summarised theorists' attempts to classify and describe metacognition. She refers to Kontos (1983) who describes knowing and conscious control of cognitive processes, and Brown, Bransford, Ferrara, and Campione, (1983) who state that knowledge about cognition and self regulation of cognition are equally important twin aspects of metacognition. Brown defined metacognition simply as referring to 'one's knowledge and control of one's own cognitive system.' (1987:67). Schraw and Dennison (1994) include a reflective component in their definition of metacognition: 'Metacognition refers to the ability to reflect upon, understand and control one's learning.' (1994:460)

An understanding of the term metacognition is made difficult because it is used to refer to two distinct areas of research: knowledge about cognition and regulation of cognition, Garofalo and Lester (1984), Brown (1987) and Schoenfeld (1990). The problem of definition has implications for methodology because the definition provides the parameters for research and a means to analyse the results of the investigation.

My synthesis of the preceding literature overview takes the form of the following definition: Metacognition refers to the *awareness individuals have of their own thinking and their ability to evaluate and regulate their own thinking*.

Figure 1 represents metacognition as it is employed in this paper. It is used as a framework and may be helpful for teaching practitioners. The model shows that there are three functions of metacognition which include: Awareness, Evaluation and Regulation of one's own thinking. Awareness and Evaluation are components of the thinking activity classified as Monitoring. Reflection is the mediating process whereby Awareness may become Evaluation and Evaluation may be transformed into Regulation of the thinking processes.

Figure 1. Model of Metacognition



The three functions of metacognition include: 1.*Metacognitive Awareness* which relates to an individual's awareness of where they are in the learning process, their knowledge about personal learning strategies and what needs to be done. 2.Metacognitive *Evaluation* which refers to judgements made regarding one's thinking capacities and limitations as these are employed in a particular situation or as self-attributes. For example, individuals could be making a judgement on the effectiveness of their thinking and strategy choice. 3.*Metacognitive Regulation* which occurs when individuals modify their thinking. They make use of their metacognitive skills to control their knowledge and thinking. They reflect on their knowledge about self and strategies (how and why they may use particular strategies). Metacognitive Regulation may include the ability to plan, self-correct, set goals and the effective use of one's own cognitive resources.

It is acknowledged that metacognition occurs within a context (such as classrooms) and that other aspects related to learning, such as students' prior knowledge, abilities, preferred ways of learning, expectations (Biggs, 1993) and volition (Corno, 1993) affect the nature of learning and thinking. These have not been discounted as important because they have not been represented on the model (Figure 1).

Methodology

The methodology used in this study was naturalistic. The data collection was primarily qualitative, deemed more suitable for aspects of life which are not directly observable. The methods used in this study were chosen after consideration of the following:

- questions of legitimacy in assessing metacognition,
- the effects of the various data collection techniques on metacognitive behavior and the
- suitability of each method when used with children.

Questions of legitimacy

There are many objections raised about the legitimacy of researching and assessing metacognition. These objections are explored and addressed in this paper. For example, Metacognition cannot be assessed because:

1. An agreement on a definition has not been achieved.
2. Researchers can only assess the consequences of metacognition.
3. The validity of self-reporting (a technique often associated with metacognitive research) is questionable (Nisbett and Wilson, 1977, Nuthall and Alton-Lee, 1995).

Garofalo and Lester (1984) suggest there are at least three reasons why the link between metacognition and mathematics performance has not been studied systematically. Firstly, they claim, mental activity is difficult to observe and analyse. If, as some psychologists suggest, people have no direct access to their mental processes, self-reports should be considered highly questionable. Increased linguistic abilities are sometimes given as a reason as to why student metacognitive awareness and control is reported to increase with age. Thorpe and Satterly (1990) share a concern for the adequacy of self-reporting measures of metacognition suggesting that results may reflect language development rather than metacognitive change.

Secondly, Garofalo and Lester (1984) argue that when self-reports are used while subjects perform a task this may stimulate metacognition rather than provide data on cognition and metacognition. Clarke (1992) also alerts us to the influence assessment has on student mathematical behavior and performance. He reports that assessment distorts the behavior it was intended to monitor.

Thirdly, Garofalo and Lester (1984) assert that as long as metacognition remains ill-defined, research in the area will attract little attention. Although the literature search results would agree with the lack of shared meaning for the term metacognition, the amount of research conducted in the field suggests that the issue has created intrigue and engaged many researchers, particularly since 1984. Nevertheless, the difficulties in assessment of metacognition in mathematics continue to exist and to thwart educators and researchers.

Methods overview

Pilot Study Sample

Questionnaires and interviews were trialled with two groups of grade six children (total n=15) in Melbourne. One group were from an outer suburban school (A, n=8). 5 of these 8 students were interviewed immediately after the questionnaire implementation. A further 7 questionnaires were given to an inner suburban cohort of children.

Questionnaires and Interviews

Likert scales and open ended questions were chosen as part of the questionnaire and interview process. Because of the difficulty of misinterpretation of questions, pretesting of the questions was conducted with a similar sample of children at another site before the pilot study. The distribution of responses from school A was similar to that of school B. The results of the teacher questionnaire have not been reported in detail in this paper.

Self-reporting

Students were asked to solve two problems in the questionnaire and then to record what they did using Likert scales (see inventory items below). The interviewed students were asked to solve another problem and report on their thinking using another set of the same Likert scales. Students were not asked to talk aloud (as in some studies, for example, Lesgold, Lajoie, Logan, and Eggan, 1990) as they solved the problems because of the possibility that verbalisation could effect, rather than accurately monitor metacognition (Clarke, 1992).

Types of mathematics tasks

Two mathematics problems were used as a basis for students' reflections on their own metacognitive thinking. The problems were non-routine and challenging because it is recognised that some students do not have to reflect to solve many classroom mathematics problems (Fortunato et al, 1991). Some mathematics tasks do not demand the use of metacognitive functions therefore careful selection was required. The tasks needed to provide an effective basis for reflection in the inventory sections of the questionnaire. Further discussion about the connection between various mathematics task types and metacognition is beyond the scope of this paper.

Inventory

The questionnaire was based around an inventory of metacognitive behaviors presented on Likert scales. The inventory items were constructed after a thorough examination of the literature and based on my definition of metacognition. This technique has been used in other studies of metacognition (Fortunato, Hecht, Kehr Tittle, and Alvarez 1991, Stacey, 1990, Goos, 1995, and Grubaugh and Speaker, Jr. 1991-2). In the search of education literature, no inventories were found for assessing the metacognition of children at the grade six level in mathematics. Most have been completed with students at tertiary or secondary levels.

The mathematics questionnaires used by Fortunato et al (1991) and Stacey (1990) were influential in the design of the questionnaire. Both were conducted with secondary school students therefore they needed to be simplified for primary age children.

Students were asked to indicate whether they always, sometimes or never did the following things when they solved mathematics problems :

a. Think about what you already know
b. Ask yourself a question about the problem
c. Think about what the problem is asking you to do.
d. Make a plan to work it out
e. Try to remember if you have ever done a problem like this before
f. Guess the answer because it is like a problem you've done before
g. Think about what you will do next
h. Go back and check your work
i. Think about a different way to solve the problem
j. Change the way you are working

Some of these items have since been modified as a result of the pilot study.

Results

The results of the questionnaires and interviews are summarised here. The findings about the research methods are considered important because the effectiveness of the instruments for gathering data on metacognition raises questions for further studies on mathematics and metacognition. More detailed results are reported in Wilson (1997).

Results about metacognition within the context of mathematics

Subjects were grouped according to the similarity of their responses to particular items. Items represent metacognitive functions as shown in the brackets.

Most students said they *always* :

a. Thought about what they already know. (awareness)
c. Thought about what the problem was asking them to do. (awareness)
d. Made a plan to work it out. (regulation)
e. Tried to remember if they have ever done a problem like this before. (awareness)

People often suggest that student reports of their behaviors merely reflect their teachers instructions and values. In this study there was little correspondence between what the interviewed teacher said she did when instructing the class in mathematics and what the students reported. Interestingly, on items where students demonstrated a range of responses, the teacher said she *never* instructed her students to ask questions about the problem (item b), that she *sometimes* asked them to think about what they would do next (item g) and that she *always* asked students to go back and check their work (item h).

Results about research methods

Student Questionnaire: When children are asked to recall decisions they make during mathematics problem solving they often do not report metacognitive thinking. Including a problem in the questionnaire provided a context for students to reflect on. Because the use of metacognition varies between contexts, in other studies, students could be asked to report on a problem just completed as well as what they usually do when they solve mathematics problems (retrospective reflection).

Student Interviews: The interviews provided opportunity for students to elaborate on responses and to follow up discrepancies between questionnaire responses on different tasks. The interviews also provided an opportunity to observe students as they solved another problem. Many student responses highlighted difficulties in reporting on one's own thinking. For example, when asked about their mathematics problem solving strategies, many students replied similarly: "Don't know", "It came automatic", "Just known before" and "Did it in my head". These sort of responses are common and frustrating for teachers who are trying to gauge student understandings and misconceptions. Sometimes they reveal a lack of metacognition and at other times demonstrate that the problem solving process has become automatic.

Some student comments offered insights into their thinking. For example, the following quote suggests that regulation of thinking may be related to the perceived importance of the task. Tests seemed to provide the motivation for encouraging action. This response followed a question about student self-assessment in mathematics.

> "Yes, occasionally I think about how I could have done it better. ... I think about my answers and if it's good and correct. If I could do it a better way. Try different ways. Even try them, sometimes. If I think it's a better way to do mathematics questions and you get better marks I'd probably write it out better-if it's like a test. If it's like normal classroom mathematics, I always try hard but I put more effort into a test."

Discussion and Implications

The findings of this small project raise many questions about the use of and assessment of metacognition in mathematics. One of the reasons why metacognition has presented such a methodological challenging research topic is because of the difficulties in clarifying what metacognition actually is. I believe the logistics of assessing metacognition in classrooms are partly based on this problem. The following key findings about assessing metacognition are made. Some suggestions for reducing the difficulties associated with assessing and researching metacognition. These have implications for teachers and teacher educators.

Key finding 1: *Existing definitions of metacognition are insufficiently precise and difficult to interpret in operational terms.* A workable model of metacognitive behavior has been provided in this paper. A framework is important for teachers interested in changing their mathematics curriculum delivery (Borkowski and Muthukrishna, 1992) and assessment techniques.

Key finding 2: *Self-reporting appears to be a valid method for researching metacognition when used in conjunction with other reliable data collection methods.* Corroborating evidence should be provided, for example through teacher observations. In classrooms a simple inventory of metacognitive behaviors, such as the one outlined in this paper, or the multi-method approach outlined below could be used. Tasks which require an achievable level of challenge are required as a context for reflections on metacognition. Where there is a mismatch between students' self-assessment and the teacher's assessment, further validation could be required.

A new multi-method approach

A new technique which is based upon the design outlined in this paper and draws on the strengths of many commonly used assessment strategies has been pretested as a result of this pilot study. The new technique has been given the term 'multi-method interview'. It includes aspects of the pilot questionnaire: verbal Likert scale, self-assessment, observation, clinical interview activity, think aloud technique (where chosen by the participant), and audio recording. This technique requires minimum preparation and little time to administer. It could be easily applied to many mathematics contexts and to a range of educational settings.

The technique would make use of the revised items on the Likert scales but the procedure would be implemented in a more creative and 'hands on' way. Subjects would be asked to have a go at solving a mathematics problem. Each metacognitive behavior item would be listed individually on a playing card (Clarke, 1989) for subjects to sort and sequence according to how they solved the problem. Cards listing cognitive behaviors and blank cards would also be provided. The resultant sequence of metacognitive behaviors suggested by subjects could be used to confirm or refute the postulated sequence of metacognitive functions (awareness, evaluation and regulation). Cards which did not apply would be discarded for later discussion about students usual metacognitive behavior when solving mathematics problems. Each card could be used to elicit information about how often students might do these things when solving mathematics problems, for example, on all problems or a few problems. Discussion could raise issues about the connection between different mathematics tasks and use of metacognitive functions.

Conclusion

It has been argued that assessment practices in mathematics must include assessment of student metacognitive thinking. Otherwise assessment procedures are capable of denying students the opportunity to use and develop metacognitive thinking which is so crucial in mathematics. If assessment of students' metacognitive thinking is not integral to mathematics teaching and learning, the improvement of student metacognition cannot be assumed: only a basic mathematics curriculum for our students can be expected.

To be attractive to practising classroom teachers, an assessment strategy for metacognition, such as the new multi-method approach outlined, must be easily integrated into mathematics instruction.

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An Overview of a Research-based Framework for Assessing and Teaching Early Number

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This paper provides an overview of a research-based framework for assessing and teaching in early number which has been used extensively in research and practice. The framework includes the following aspects: arithmetical strategies, forward and backward number word sequences, numeral identification, base-ten strategies, combining and partitioning, subitising and spatial patterns, temporal sequences, finger patterns, and quinary-based strategies. Five of these aspects are presented in tabular form as stages or levels. Also described are guiding principles for studying early number knowledge.

The purpose of this paper is to provide an overview of the Learning Framework in Number (LFIN), a research-based framework for the assessment and teaching of number in the early years of school. The LFIN was developed during 1996, for the NSW Department of School Education's Count Me In (CMI) project, and has been used extensively by all participants (including teachers, district consultants and researchers) in CMI and Count Me In Too (CMIT), a project subsequent to CMI. Since 1992, LFIN and its antecedent forms have constituted the theoretical framework for the Mathematics Recovery Program (Wright, 1994b; Wright, Cowper, Stanger & Stewart, 1995; Wright, Stanger, Cowper, & Dyson, 1996) and several empirical studies (eg Wright, 1994a). Thus the LFIN has been used extensively in research and practice. Finally, development of the LFIN drew extensively on recent research in early number learning.

Table 1
Organisation of LFIN.

Part A	A1 — Early Arithmetical Strategies
	A2 — Base-Ten Arithmetical Strategies
Part B	B1 —Forward Number Word Sequences (FNWSs)
	B2 —Backward Number Word Sequences (BNWSs)
	B3 — Numeral Identification
Part C	C1 — Combining and Partitioning
	C2 — Spatial Patterns and Subitising
	C3 — Temporal Sequences
	C4 — Finger Patterns
	C5 — Quinary-based Strategies
	C6 — Early Multiplication, Division and Fraction Knowledge

Organisation of the LFIN.

The LFIN is organised into three parts which span 11 aspects of early number learning (Table 1). All aspects are considered to be of potential significance when one is attempting to comprehensively understand and document a young child's number knowledge. For each aspect, the LFIN sets out descriptions of children's strategies or knowledge and a range of exemplary tasks or guidelines for assessment. Because of space limitations, detailed descriptions of these are not provided in this overview. Each of the five aspects in Parts A and B is presented in tabulated form as stages or levels. These tabulated forms — referred to as models — facilitate profiling of a child's early arithmetical knowledge. In four of the models the initial level (or stage) is labelled "Level 0 (or Stage 0)" and is also labelled "emergent". The description at Level 0 is typically in terms of the child not being able to perform at Level 1. Each of the 11 aspects can be regarded as viewpoints from which to observe young children's strategies in number and to document their number knowledge. These aspects should not be regarded as necessarily distinct from one another. Rather, they are potentially overlapping. Thus observations of two or more aspects may reveal similar features of a child's early number knowledge. As well, the extent of overlap in this sense, is likely to vary widely across young children.

Guiding Principles for Studying Early Number Knowledge

Development of the LFIN was informed by an extensive and on-going research program in early mathematics. This research program includes use of a distinctive approach to observing and documenting young children's arithmetical knowledge and strategies, which has been developed by the author and which has its origins in the constructivist research methodology developed by Steffe and colleagues (eg Steffe & Cobb, 1988). Outlined in the following section are some of the guiding principles and associated terminology of this approach. The use of this approach has provided an extensive empirical base for development of the LFIN. This base consists of videotaped records of individual interviews during which children attempt to solve mathematical tasks which are problematic for them.

Observation and assessment : Underlying the LFIN is a belief that, in early arithmetical learning it is very important to understand, observe and take account of children's knowledge and strategies. Children's early arithmetical knowledge varies greatly and their strategies are multifarious. Thus, across children, early arithmetical knowledge is characterised by both commonalities and diversity. It is insufficient to think that all children's early arithmetical knowledge develops along a common developmental path (Denvir & Brown, 1986). For example, one important factor in a particular child's developmental path it is believed, relates to the nature of the settings in which the child's prior learning has occurred. As well, children who may appear to an observer to be in the same setting or learning situation will construct the situation idiosyncratically and thus different kinds of learning are likely to occur. The child's process of constructing arithmetical knowledge can be thought of in terms of progression or advancement. Children reconstruct or modify their current strategies and doing so is nothing more or less than progression, advancement or learning. Given this, it is useful to consider the notion of the relative sophistication of children's strategies. For example, the child who has no means of working out nine plus three other than counting out nine counters from one, counting out three counters from one, and then counting all of the counters from one to 12, is using a less sophisticated strategy than the child who ignores the counters and says nine plus three is the same as ten plus two, and I know that is 12 without counting.

Settings, tasks, strategies and procedures : In the LFIN, the term "task" is used as the generic label for particular problems or questions presented to a child in the interview, while "setting" is used as the generic label for physical situations used by the teacher in posing a task. Examples of tasks are: given two covered collections, asking the child how many in all; asking the child to say the number words from "twenty" to "ten"; asking the child to hold up four fingers on each hand and say how many in all. Examples of settings are: a row of numeral cards, two piles of counters, a Tens Frame, a Hundreds Chart. A "strategy" is a particular method by which a child solves a task. Finally, a "procedure" is a part of a strategy, and strategies consist of one or more procedures. As well, specific procedures may be part of differing strategies, and specific procedures may be relevant to more than one of the aspects in the LFIN. For example, the procedure of raising fingers in turn to keep track of counting-on is relevant to the aspects of arithmetical strategies and fingers and finger patterns.

Presenting assessment tasks : A necessary part of determining a child's strategies is to present the child with tasks or situations which are problematic for her or him. In such situations the child's goal we assume, is to resolve the difficulty, ie to solve the problem as he or she constructs it. In doing so the child uses a current strategy or may construct a strategy which is novel for that child. It is not essential that the tasks presented to children be what adults might label as "real-world" or everyday situations.

For example, it can be just as useful and productive to pose a problem about numbers, number words or numerals per se, as it is to pose a problem about an everyday situation. Observations of children's solutions to a range of arithmetical problems lead to an understanding of the range, diversity and progression of children's strategies. With this knowledge one can determine a profile of arithmetical knowledge for individual children.

Eliciting strategies **:** A final but important point is that children frequently use strategies that are less sophisticated that those of which they are capable. This may happen for one or more of several reasons, eg (a) it may be easier and although it may take more time, this may not be of concern to the child; or (b) some feature of the child's thinking immediately prior to solving the current task may focus the child's attention on a less sophisticated strategy. Thus an important challenge for the teacher as observer and diagnostician is to attempt to elicit the child's most sophisticated strategy. This is crucial to gaining an adequate understanding of the child's current knowledge. Eliciting a strategy is not necessarily the same as asking the child to describe their strategy, after he or she has solved a problem. Close observation and informed reflection while interviewing are crucial to understanding a child's strategy. Asking a child to describe a strategy is sometimes useful but equally, can be off-putting for the child particularly if this becomes a routine directive. As well, when asked to describe a strategy just used, a child may unwittingly or intentionally describe a strategy different from the one used.

Aspects of the Learning Framework in Number

Following are descriptions of all but one (ie early multiplication, division and fraction knowledge) of the 11 aspects listed in Table 1, and each of the first five of these is presented in tabular form.

Table 2.
Model for Development of Early Arithmetical Strategies

Stage 0: Emergent Counting. Cannot count visible items. The child either does not know the number words or cannot coordinate the number words with items.

Stage 1: Perceptual Counting. Can count perceived items but not those in concealed collections. This may involve seeing, hearing or feeling items.

Stage 2: Figurative Counting. Can count concealed items but counting typically includes what adults might regard as redundant activity. For example, when presented with a collection partitioned into two parts (both screened), told how many in each part and asked how many counters in all, the child will count from "one" instead of counting on.

Stage 3: Initial Number Sequence. Child counts-on rather than counting from "one", to solve addition or missing addend tasks. The child may use a count-down-from strategy to solve removed items tasks (eg 17-3 as 16, 15, 14 - answer 14) but not count-down-to strategies to solve missing subtrahend tasks (eg 17-14 as 16, 15, 14 - answer 3).

Stage 4: Intermediate Number Sequence. The child counts-down-to to solve missing subtrahend tasks (eg 17-14 as 16, 15, 14 - answer 3). The child can choose the more efficient of count-down-from and count-down-to strategies.

Stage 5: Facile Number Sequence. The child uses a range of what we refer to as non-count-by-one strategies. These strategies involve procedures other than counting by ones but may also involve some counting by ones. Thus in additive and subtractive situations, the child use strategies such as compensation, using a known result, adding to ten, commutativity, subtraction as the inverse of addition, awareness of the "ten" in a teen number.

A1: Early Arithmetical Strategies. This aspect is based on extensive research by Steffe and colleagues (eg Steffe & Cobb, 1998; Steffe, 1992a). The model for this aspect appears in Table 2. Included in this aspect are:

(a) Initial counting strategies for establishing numerosity, or solving additive or subtractive problems, ie counting from one with items perceived, and counting from one without items perceived.

(b) Advanced strategies involving counting-by-ones, eg counting-up-from and counting-up-to — collectively referred to as counting-on, counting-down-from, and counting-down-to.

(c) Non-count-by-one strategies, ie cases where at least part of the strategy involves procedures that do not involve counting-by-ones, eg commuting an addition; using addition to work out a subtraction; using easier additions which are habituated, eg doubles; using known combinations to five or ten and knowledge of the tens structure of teens or of numbers from 20 onward.

A2: Base-Ten Arithmetical Strategies. Around the time they attain Stage 3 on the Model of Early Arithmetical Strategies children typically develop beginning knowledge of the tens and ones structure of the numeration system. Of course children can and should solve addition and subtraction tasks involving two-digit numbers long before they develop knowledge of the tens and ones structure. For children who have attained Stage 5, development of knowledge of the tens and ones structure becomes increasingly important. Table 3 outlines a progression of three levels in children's development of base-ten arithmetical strategies (Cobb & Wheatley, 1988).

Table 3
Model for Development of Base-Ten Arithmetical Strategies

Level 1: Initial Concept of Ten. The child does not see ten as a unit of any kind. The child focuses on the individual items that make up the ten. In addition or subtraction tasks involving tens, children count forward or backward by ones.

Level 2: Intermediate Concept of Ten. Ten is seen as a unit composed of ten ones. The child is dependent on re-presentations* of units of ten such as hidden ten-strips or open hands of ten fingers. The child can perform addition and subtraction tasks involving tens where these are presented with materials such as covered units of tens and ones. The child cannot solve addition and subtraction tasks involving tens and ones when presented as written number sentences.

Level 3: Facile Concept of Ten. The child can solve addition and subtraction tasks involving tens and ones without using materials or re-presentations of materials. The child can solve written number sentences involving tens and ones by adding or subtracting units of ten and ones.

* A re-presentation can be thought of as a mental replay (ie in reflection) of a prior experience that is distinct from and separated in time from the experience itself (See von Glasersfeld, 1987).

Part B

The three aspects in Part B are concerned with children's facileness on important specific aspects of children's early arithmetical knowledge: forward number word sequences (FNWSs); backward number word sequences (BNWSs); and numeral identification. There has been little systematic study of children's strategies relating to these aspects and thus little is known about the idiosyncratic nature and diversity of their strategies (Wright, 1998).

B1 & B2: FNWSs and BNWSs. From the point of view of fully understanding children's early arithmetical knowledge, it is useful to construe as distinct, the two aspects concerned with number word sequences, ie forward and backward. Models associated with these aspects are shown in Tables 4 and 5.

Table 4
Model for Forward Number Word Sequences (FNWSs)

Level 0: Emergent FNWS. The child cannot produce the FNWS from "one" to "ten".

Level 1: Initial FNWS up to "ten". The child can produce the FNWS from "one" to "ten". The child cannot produce the number word just after a given number word in the range "one" to "ten". Dropping back to "one" does not appear at this level. Children at Levels 1, 2 and 3 may be able to produce FNWSs beyond "ten".

Level 2: Intermediate FNWS up to "ten". The child can produce the FNWS from "one" to "ten". The child can produce the number word just after a given number word but drops back to "one" when doing so.

Level 3: Facile with FNWSs up to "ten". The child can produce the FNWS from "one" to "ten". The child can produce the number word just after a given number word in the range "one" to "ten" without dropping back. The child has difficulty producing the number word just after a given number word, for numbers beyond ten.

Level 4: Facile with FNWSs up to "thirty". The child can produce the FNWS from "one" to "thirty". The child can produce the number word just after a given number word in the range "one" to "thirty" without dropping back. Children at this level may be able to produce FNWSs beyond "thirty".

Level 5: Facile with FNWSs up to "one hundred". The child can produce FNWSs in the range "one" to "one hundred". The child can produce the number word just after a given number word in the range "one" to "one hundred" without dropping back. Children at this level may be able to produce FNWSs beyond "one hundred".

Table 5.
Model for Backward Number Word Sequences (BNWSs)

Level 0: Emergent BNWS. The child cannot produce the BNWS from "ten" to "one".

Level 1: Initial BNWS up to "ten". The child can produce the BNWS from "ten" to "one". The child cannot produce the number word just before a given number word. Dropping back to "one" does not appear at this level. Children at Levels 1, 2 and 3 may be able to produce BNWSs beyond "ten".

Level 2: Intermediate BNWS up to "ten". The child can produce the BNWS from "one" to "ten". The child can produce the number word just before a given number word but drops back to "one" when doing so.

Level 3: Facile with BNWSs up to "ten". The child can produce the BNWS from "one" to "ten". The child can produce the number word just before a given number word in the range "one" to "ten" without dropping back. The child has difficulty producing the number word just before a given number word, for numbers beyond ten.

Level 4: Facile with BNWSs up to "thirty". The child can produce the BNWS from "one" to "thirty". The child can produce the number word just before a given number word in the range "one" to "thirty" without dropping back. Children at this level may be able to produce BNWSs beyond "thirty".

Level 5: Facile with BNWSs up to "one hundred". The child can produce BNWSs in the range "one" to "one hundred". The child can produce the number word just before a given number word in the range "one" to "one hundred" without dropping back. Children at this level may be able to produce BNWSs beyond "one hundred".

B3: Numeral Identification. Learning to identify, recognise and write numerals may be regarded important in early literacy development but is also important in early arithmetical development. The term 'identify' is used here with precise meaning, ie to state the name of a displayed numeral. The complementary task of selecting a named numeral from a randomly arranged group of displayed numerals is referred to as 'recognising'. This corresponds with typical use of these terms in psychology and early literacy. Table 6 outlines a progression of four levels in children's development of numeral identification.

Table 6.
Model for the Development of Numeral Identification
Level 0: Emergent Numeral Identification.
Cannot identify some or all numerals in the range "1" to "10".
Level 1: Numerals to "10"
Can identify numerals in the range "1" to "10".
Level 2: Numerals to "20"
Can identify numerals in the range "1" to "20".
Level 3: Numerals to "100"
Can identify one and two digit numerals.
Level 4: Numerals to "1000"
Can identify one, two and three digit numerals.

Part C

C1: Combining and Partitioning. Counting strategies are an important aspect of children's early arithmetical knowledge. Nevertheless, children also develop knowledge of simple addition combinations, eg involving two addends in the range one to five, which does not rely on counting. Children learn to provide answers almost immediately to questions such as three plus three, using procedures that do not involve counting-by-ones. Arithmetical knowledge of this kind has been labelled "automatised" or "habituated". Recent research provides strong indications that teaching children to habituate simple addition facts through combining and partitioning of small numbers can significantly facilitate development of advanced arithmetical strategies, ie non-count-by-one strategies (eg Cobb, McClain, Whitenack, & Estes, 1995).

C2: Spatial Patterns and Subitising. Experiences with spatial patterns can be important in early number (eg Bobis, 1996). Children's strategies related to this aspect of early number arise in settings involving spatial configurations of various kinds, eg domino patterns, pairs patterns, Tens Frame, playing card patterns, regular plane figures and random arrays. Subitising is "the immediate, correct assignation of number words to small collections of perceptual items" (von Glasersfeld, 1982, p. 214). As explained by von Glasersfeld, when a young child for example, says "three" in response to a briefly displayed spatial array, the child may be doing no more than recognising and naming a spatial arrangement. One cannot assume that this child has a concept of "three". Nevertheless being able to name spatial arrays in this way is an important ability in early number (eg Bobis, 1996). No doubt at some point the child will see a correspondence between the name of the array, ie "three" and the last number word when they count the dots in the array, eg "one, two, three!". In early number there is a range of instructional settings for which spatial pattern or spatial arrangement seems to be a dominant feature. These include cards with random or irregular arrays, the Tens Frame, plane figures (triangles, squares etc.), and the various regular patterns (eg domino patterns for 1 to 6, pairs patterns for 1 to 10, and the patterns on playing cards).

C3: Temporal Sequences. Temporal sequences (von Glasersfeld, 1982, p. 201-204) involve events that occur sequentially in time, eg sequences of sounds and movements. Sequences of sounds may be rhythmical, arrhythmic or monotonic. Instructional settings in which the child's task is to count or copy a sequence of sounds or count a sequence of movements are considered likely to enhance early arithmetical knowledge. There has been little systematic study of children's strategies associated with temporal sequences. Experience has shown that, as a general rule, children are not as facile at counting temporal sequences as they are at counting items occurring in spatial sequence, eg rows of counters or dots. For example, in contrast to counting a row of counters, when counting a sequence of sounds the child has no control of the speed of perception of the individual items. Children will count slow monotonic sequences of sounds and sequences of movements similarly to the way they count a row of items, ie to coordinate a number word in sequence with their perception of each item. When counting rapid, rhythmical sequences of up to six sounds three strategies may be used: (a) to count the individual sounds from one as they occur; (b) to recognise the pattern in terms of its number of beats and thus answer without counting; and (c) to mentally replay or re-present the pattern, after its completion, and count the number of beats in re-presentation.

C4: Finger Patterns. Using one's fingers is very prominent in early arithmetic. There are variations in ways children move and make patterns on their fingers. For example, some will begin with five fingers raised, and lower fingers in turn while others begin with the five lowered and curled, and raise them in turn. Fingers are used in a range of ways and with varying levels of facility and sophistication. The current author has used the label "counting forward from one three times" for a prominent strategy involving perceptual counting, used to solve additive or subtractive tasks involving numbers up to 10. In the case of subtraction for example, a child might sequentially raise fingers while counting from one to the given minuend, then sequentially lower fingers while counting from one to the given subtrahend and then count the remaining fingers from one. In contrast, a strategy involving simultaneous raising of fingers to symbolise a number is more advanced. Another example of a more advanced use of fingers is the sequential raising of fingers to keep track of counts when using an advanced counting-by-one strategy (see earlier). This strategy, like the second last mentioned, also involves knowledge of a finger pattern and unlike the first mentioned, does not involve counting fingers sequentially to establish numerosity.

C5: Quinary-based strategies. These strategies involve using the number "five" as a base and arise in instructional settings involving the arrangement of items in fives, eg Arithmetic Rack (Gravemeijer, 1994, pp. 182-5) and Tens Frame, and also settings where use of fingers is prominent. Typically in these settings, the number "ten" is also a base. In early number there is a major potential advantage associated with "five" being used as an additional base along with "ten". Use of the base "five" has the potential to greatly reduce reliance on counting-by-ones. In other words, promoting the use of base-five in early number may well result in earlier development of facile arithmetical strategies, eg non-count-by-one strategies (eg Gravemeijer, 1994; Gravemeijer, Cobb, Bowers, & Whitenack, in press).

C6: Multiplication, division and fractions. As topics in early number these would reasonably be viewed as more advanced than addition and subtraction (eg Hunting, Davis & Pearn, 1997; Steffe, 1992b). Nevertheless, children in the early years (K-2) can and do reason multiplicatively (eg Mulligan & Mitchelmore, 1997) and in terms of fractions (eg Hunting & Sharpley, 1988). In its initial form the LFIN did not address this aspect but in its revised form it includes a section written by Mulligan (1997) focusing on young children's strategies in multiplication and division.

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Language, mathematics and social disadvantage: a Bourdieuian analysis of cultural capital in mathematics education

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What do students need to know if they are to be constructed as effective learners of mathematics? This question provides the stimulus for this paper where the question is rephrased to: "What do students need to know in order to operate in a manner which is acceptable in the mathematics classroom?" Such a question is not without political implications and so needs to be extended to include questions about the consequences of participation in mathematics classroom. It is widely recognised that success in mathematics is not random, but rather falls into quite distinct patterns whereby students from certain social groups are more likely to be successful in the study of mathematics than others. The focus in this paper is the examination of why students from socially-disadvantaged backgrounds are less likely than their middle-class peers to be successful in the study of mathematics. The paper uses wo key notions. The first is the classroom interaction patterns noted by Lemke (1990) in which students must be conversant to be able to participate effectively. This knowledge becomes a form of cultural capital (Bourdieu, 1983) which can be transfered later to academic success.

Language as a Form of Capital

The framework used for this paper supports a critical perspective of educational successful and it is through the concepts developed by Pierre Bourdieu that educational success is framed. Of particular importance to this paper are Bourdieu's notions of cultural capital and habitus. Bourdieu (1983) has argued for the notion of cultural capital which exists in a number of forms. Of particular import to this paper is the notion of linguistic capital in which language is seen as a form of capital which can be exchanged for other forms of capital - social, economic or cultural. Using this framework, I argue that students enter the mathematics classroom from a range of socio-cultural backgrounds. Within these contexts, students whose socio-cultural background is congruous with that of the culture represented in and through the practices embedded within the mathematics classroom - including linguistic practices - are more likely to be constructed as successful students.

Bourdieu (in Bourdieu & Wacquant, 1992) argues that access to legitimate language, in this case mathematics, is not equal and that linguistic competence is monopolised by some. In considering the case of mathematics, this suggests that access to the discourses and discursive practices of mathematics is differentially accessible. For those students who enter the mathematics classroom with a competence in the discursive practices, access to mathematics is made more easily. Simultaneously, such students are more likely to be constructed as successful students based on the teacher's judgement of their ability. Within this context, language background is a form of capital which can be converted to academic reward.

Linguistic competence – or incompetence – reveals itself through daily interactions. Within the mathematics classrooms, legitimate participation is acquired and achieved through a competence in the classroom dialogic interactions. Students must be able to display a discursive competence which incorporates a linguistic competence, an interactional competence along with a discursive competence if they are to be seen as competent learners of mathematics. Classroom interactions are imbued with cultural components which facilitate or inhibit access to the mathematical content. To gain access to this knowledge, students must be able to render visible the cultural and political aspects of the interactions. Bourdieu (in Bourdieu & Wacquant, 1992) argues that

> Linguistic competence is not a simple technical ability, but a statutory ability. ...what goes in verbal communication, even the content of the message itself, remains unintelligible as long as one doe not take into account the totality of the structure of the power positions that is present, yet invisible, in the exchange. (p. 146)

From their early years, students are located within family structures and practices which will facilitate the development and embodiment of particular cultural features, least of which is language. For these students, the embodiment of their cultural background into what Bourdieu refers to as the habitus, predisposes them to think and act in particular ways. This embodiment of culture includes a linguistic component. Students whose linguistic habitus is congruent with that of the discursive practices represented in mathematics classrooms are more likely to have greater access to the knowledge represented in and through such practices.

From this perspective, language must be understood as the linguistic component of a universe of practices which are composted within a class habitus. Hence language should be seen to be considered as another cultural product - in much the same ways as patterns of consumption, housing, marriage and so forth. When considered in this way, Bourdieu proposed that language is the expression of the class habitus which is realised through the linguistic habitus.

> Of all the cultural obstacles, those which arise from the language spoken within the family setting are unquestionably the most serious and insidious. For, especially during the first years of school, comprehension and manipulation of language are the first points of teacher judgement. But the influence of a child's original language setting never ceases to operate. Richness and style of expression are continually taken into account, whether implicitly or explicitly and to different degrees. (Bourdieu, Passeron, & de saint Martin, 1994a) p. 40

The intersection of language, mathematics and culture has been recognised in both the literature and by teachers. Most often it is seen through deficit models where the linguistic background of the students is seen to be poor and hence contributes to the students capacity to participate in classroom activities. Comments such as those below are common explanations of the role of language in mathematics learning and teaching.

> T You talk to the preschool teachers. Some of the kids can't even speak properly....They can't communicate properly to get their message across. There are no sentences. Saying one word things to get something. Things like boundary, we had ummm., out of bounds, you know you're in trouble. They know that. Out of bounds, they get booked, no trouble with that. We were doing mapping and I said where is the boundary between such and such and such and such and I had all these blank faces and I had to explain boundary. I said what does boundary mean. They didn't know what boundary meant. I said "What does out of bounds mean?". Their answer was you get in trouble. They didn't understand that it is the boundary.

In contrast, the model proposed in this paper is based on a difference model and suggests that the linguistic habitus of the student will have substantial impact on his/her capacity to make sense of the discursive practices of the mathematics classroom and hence their subsequent capacity to gain access to legitimate mathematical knowledge along with the power and status associated with that knowledge. The processes through which the schooling procedures are able to value one language and devalue others must be systematically understood. Through this process, we will better understand how mathematical pedagogy both inculcates mathematical knowledge and imposes domination. In order to understand how the linguistic practices of the mathematics classroom position and hinder the effective participation of some students, the notion of "mathematical pedagogy as a text" is proposed.

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Mathematics as a Text

Gore (1990) has argued that pedagogy can be seen as a text which can be read and interpreted by the reader. Texts can be read in a multiple of ways, so that the student entering the mathematics classroom will be required to read, interpret and make sense of what transpires in the mathematics classroom - not only of the mathematical content, but also the pedagogical approaches within which the content is relayed. To be able to read these texts, the students must have some linguistic competence in the reading of social texts and discursive practices.

The interactions that occur within the classroom have been subjected to ethnomethodological approaches and have been found to have highly ritualised components with clearly identifiable discursive practices (Lemke, 1990; Mehan, 1982a). They argue that these components are not explicitly taught but are embedded within the culture of the classroom. The highly ritualised practices of classroom interactions can be seen in the types of interactions which occur across the various phases of the lesson. For example, the most common form of interaction consists of a practice in which the teacher initiates a question, the students respond and the teacher evaluates that response which Lemke (1990) refers to as "triadic dialogue". Lemke (1990) argues that this practice allows teachers to keep control of the content and flow of this phase of the lesson along with social interactions. While Lemke focussed on the science classroom, the style and purpose of this interaction can be just as readily applied to the mathematics classroom. This practice is not made explicit to students, rather it must be learnt through implicit means. To participate in the classroom interactions effectively, students must have knowledge – either intuitive or explicit – of these unspoken rules of interaction.

Furthering the work of Lemke, Mehan (1982b) has identified three key phases of a lesson - the introduction, the work phase and the concluding/revision phase. In each of these phases there is a shift in the power relations between the students and teacher which permits different forms of interactions to occur (Mehan, 1982b; Schultz, Florio, & Erickson, 1982). For the purposes of this paper, it is my intention to discuss the introductory phase only.

Mehan (1982) argues that during the introductory phase of the lesson, the teacher maintains tight control over the students, initially to ensure that the students are ready for the content of the lesson. Once control has been established and attention gained, the lesson can then proceed. Triadic dialogue is commonly observed in this phase in order to keep control of the academic content of the lesson and the control of the students. Dialogue between students and between teacher and students is not generally part of this phase. If the teacher initiates a question but the student is not able to respond, it is not appropriate for students to express their lack of understanding since this will interrupt the flow of the phase. If there is a misunderstanding or lack of understanding, it is more appropriate for this to be voiced in the work phase of the lesson.

What is lacking from this corpus of knowledge of classroom interaction is the failure to recognise that these interactions recognise a particular linguistic form which will be more accessible to some students than others. In this sense, the interactions within the classroom can be considered as another cultural product which is more familiar to some students and not others. The linguistic habitus of the students will or hinder a students capacity to render visible the mathematical content embedded in the pedagogic action. The

> ...such cues [IRE] are not necessarily "understood" by all participants, but they are certainly part of the "functional conflict" between dominant and dominated languages in (and out of) educational settings. (Collins, 1993, p.131)

In the previous sections, I have drawn on the work from a number of traditions – theoretical and methodological – and have proposed that the social background of the student will the construction of a particular linguistic habitus. The field of mathematics, having its own regulatory discourses and discursive practices, will recognise and value some linguistic practices and not others. These practices are socially biased.

Classroom Interactions

An ethnography of two classrooms was undertaken in which mathematics lessons were videotaped. The two classrooms were located in socially divergent sites - one an independent school which serves a middle- to upper-class clientele (Angahook). The other classroom was in a state school serving a predominantly working-class clientele (Connewarre). The classrooms were in the second last year of primary school and most students were approximately 10-11 years old. The video-taped lessons were transcribed and analysed. Extracts from one of the lessons from each classrooms will be used as examples for this paper.

Angahook

This school serves a middle- to upper-class client group. The mathematics teaching, learning, assessment and curriculum are relatively conservative with a strong emphasis on rote learning, preparation for examinations and teacher-directed pedagogy. The class sizes are small with only 12 students in classroom observed. In the lesson presented here, the students were undertaking an activity from the Mathematics Curriculum Teaching Package. Prior to the extract shown, the teacher (Helen) has used a number of short mental arithmetic tasks. The following is the introduction to the lesson.

T: You are asked to judge the diving for the Olympics, you will need to know the degree of difficulty because what if someone did just a plain dive and did it perfectly and got full marks for it and what if someone else did a triple somersault, back flip, side swinger double pike and knocker banger and only got half marks for ti because they entered the water and made a bit of a splash. Is that fair?
C: No
T So we have to talk about degrees of difficulty. What do you think that means? What does that actually mean? Robert?
B You have to add a bit more to the score because of the degrees of difficulty.
T Good boy. Yes, good. Daniel?
B Well the performance of their dive, how they dive and well like they might have a very good dive and make a very big splash and may even get off
T Right, good. OK you are on the right track. What do you want to say about degree of difficulty Cate?
G How hard it is?
T How hard it is. Tom what would you like to say about degree of difficulty? That's not a word we use much in our everyday language..... degree of difficulty.
B The percentage of how hard it is
T Good. Because you're focussing on the word degree though aren't you. So a really hard dive. Now you can see on this sheet they're talking about DD which is short for degree of difficulty and a really hard dive. What would be a really hard dive? What would be the highest number for a degree of difficulty be? Have a look at your sheet. Try and work out the degree of difficulty. Vicky?

From this extract it can be seen that the teacher follows the triadic dialogue identified by Lemke. The teacher retains control of the content and interactions through the use of the three phases of interactions. Using this approach she is able to control the flow of the lesson as can be seen in the last interaction where Tom has mentioned "percentages" which she then takes as a cue for linking percentage and degree in a way which suits her purposes.

Examining the flow of the interactions indicates that there is a complicit agreement between the teacher and students to participate in the interactions. There are no transgressions or challenges to the teacher's authority. This allows for substantive content to be covered.

The teacher is able to maintain control over both the form and content of the lesson and the students through a mutual compliance with the implicit rules by both the students and the teacher. She has used Triadic Dialogue to structure the interactions and students infrequently transgress the rules. This allows her to retain the focus of the lesson and in so doing, the students are exposed to a significant amount of mathematical knowledge that is embedded in that dialogue. The teacher's capacity to deliver the lesson in this way allows for her to use a very rich mathematical language as she discusses the mathematical content. In other words, the students are exposed to mathematical language and concepts in a style which takes for granted their linguistic background. The work of Brice-Heath (1982) has shown that middle-class students are more likely to be familiar with these forms of school interactions due to their similarity with the linguistic patterns of the home environment. This familiarity has facilitated a linguistic habitus which is similar to that of the formal mathematics classroom and hence permits access to the codes and signifiers of school mathematics.

Connewarre

Connewarre is a large government school which is located within a large housing commission estate. The clientele of the school is predominantly working class with many of the parents receiving government support. The classrooms are smaller than Angahook with approx 25-30 students in each class. The teacher introduces the mathematics lessons with problem solving activities which the students undertake as small groups. They are able to be physically involved in the activities and it is not uncommon for the students to draw on the carpet with chalk to represent the task or physically construct the problem. The mathematics has a strong emphasis on real life situations. The following extract is the introduction to a lesson in which the teacher has drawn a net on the board which the students will have to draw onto card and then construct. Students are then required to develop a number of nets for nominated prisms.

T So if I put those together we start talking more about a shape I am talking about. It's sort of a rectangle on the sides, all the way round but you don;t call it is a rectangle, because a rectangle is jus the flat surface. What do you call the whole thing if that was one whole solid shape. What do you call that?
C A cube
T He said a cube. Don't call out please.
C A rectangular rectangle
T You're on the right track
C A 3D rectangle
T Three dimensions, technically I suppose you're right.
C A rectangular
It's a rectangular something. Does anyone know what it is called?
C A parallelogram
T Put your hand up please.
C [unclear]
T No
More calling out

T I guess you could have a rectangular parallelogram, but no. A rectangle is a special parallelogram.,

C A rectangular oblong

T The word we are looking for is prism

C Yeah that's what I said

T Say the word please

Cs Prism

T Not like you go to jail "prison", that's prison. Excuse me, could you return those please.

[calling out]

T So one thing that we think about with rectangular prisms and that this shape on here is, excuse me...Now you can leave them down please. You need a little bit of practice at lunch because you can't stop fiddling. This shape here is drawn out on the graph, this grid here [net for a rectangular prism]. We 're going to try and do the same thing. Draw the shape and then cut it out. If you look at the shape, it's made up of rectangles and squares.

In this extract, it indicates that the flow of the lesson and content is hampered by the challenges to the teacher's authority. The triadic dialogue does not serve the same purpose as noted by Lemke (1990) and found at Angahook. The field of mathematics education has particular unspoken rules of interaction which have not been appropriated by the students at Connewarre, or may be resisted by the students.

There are many transgressions of the implicit rules of classroom interactions. The flow of the lesson is fragmented as students challenge the teacher's control for the floor and content of the lesson.

The linguistic habitus of the student implies a propensity to speak in particular ways which, as can be observed in the case of the interactions in this extract, works to exclude students from the mathematical content. The students are not as competent in the linguistic exchanges of the mathematical interactions as their middle-class peers thereby marginalising them in the process of learning. The teaching of mathematics in this way tacitly presupposes that the students will have the discursive knowledge and dispositions of particular social groups, namely the middle-class. The students are not as complicit in the classroom practices and in so doing are being excluded from active and full participation in the mathematics of the interactions. In this way, students have been exposed to the symbolic violence of formal education.

Conclusion

Using data from mathematics classrooms, Voigt (1985, p. 81) has argued, "The hidden regularities, the interaction patterns and routines allow the participants to behave in an orderly fashion without having to keep up visible order" so the idea is far from new. However, what I have sought to uncover using an interactionist approach is the ways in which some students are able to gain access to mathematical content and processes more readily than others. I have proposed that one subtle and coercive way is through the linguistic habitus of the students and the practices of classroom interactions whereby some students enter the formal mathematics classrooms with a habitus that is akin to that which is valorised within that context. These students will be able to participate more effectively and efficiently than their peers for whom the patterns of interaction are foreign to their habitus, thereby making the habitus a form of capital which can be exchanged for academic success within this context.

The predominantly implicit codes of curriculum and classroom interactions take as a given that students will have a familiarity with the legitimate linguistic practices of the mathematics classroom, but neither curriculum nor pedagogy render that language visible. gaining access to mathematical knowledge is facilitated, or hindered, but a match or mismatch of codes. Rather than perceive this a function of language deficiency, but as systemic through which the dominant classes are able to maintain control:

> pedagogies that tacitly select the privileged and exclude the underprepared are not regrettable lapses; they are systemic aspects of schooling systems serving class-divided societies. (Collins, 1993, p.121)

The linguistic habitus of the middle-class students positions them so that their possession of knowledge of what constitutes "appropriate" classroom linguistic exchanges is similar to that which the system values thus allowing them to participate effectively in classroom practice. Alternatively, the linguistic habitus facilitates the appropriation what the system offers. The dispositions, as per the linguistic habitus, of each of the classes have facilitated or hindered their acquisition of mathematics. The linguistic habitus is differentially valued within the mathematics classroom so that for some students the linguistic code with which they are familiar and use within the classroom becomes a form of capital which can be exchanged for other cultural goods - in this case, grades and the subsequent academic success conveyed to the individual thereby reinforcing the claim that "The more distant the social group from scholastic language, the higher the rate of scholastic mortality (Bourdieu, Passeron, & de saint Martin, 1994b, p.41).

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Mathematical saturation within workplace contexts

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In this paper I propose that the literature on situated numeracy has an important role in the reconceptualisation of mathematics education in general and workplace learning in particular. There exists a substantive literature which documents the idiosyncratic knowledge and procedures used by participants *in situ* to resolve tasks. This literature has been powerful in challenging the orthodoxies within the field of mathematics education whereby the power of mathematics comes from its capacity to be applied across a wide range of contexts. However, this situatedness of mathematics must be considered in conjunction with the degree of mathematics employed within a context. To this end, I propose and develop the notion of "mathematical saturation" which permits the breaking of the dichotomy of school-mathematics and non-school mathematics, and in its place proposes a continuum in which the degree of "pure" mathematics is considered to be a critical element. Using the data collected from a number of worksites, it will be argued that situated numeracy challenges notions of transferability but must be considered alongside the degree of formal mathematics which is used within that context.

Situated Numeracy

Within the field of mathematics education, challenges have been proposed which destabilise the orthodoxy guiding significant practices and discourses. These challenges have come from the discourses which support notions of numeracy being situational and hence highly determined by and determining the context within which the mathematics is used. Such challenges have had significant impact in the theorisation of the role and purposes of mathematics in out-of-school contexts. As such, there are important implications for teaching and learning practices when considered in conjunction with the current discourses on key competencies.

Over the past decade, the work broadly considered "situated numeracy" has brought to the fore the processes, knowledge and skills used in non-school contexts and has effectively challenged the assumption that mathematical knowledge and skills can transfer from school to non-school contexts. In contrast, situated numeracy has highlighted the highly contextual strategies used by individuals solving everyday tasks. The mathematics and problem solving strategies used in everyday contexts are substantially different from those of the formal school mathematics and in most cases, are often far more effective. This literature has posed serious challenges to the approaches underpinning much of contemporary mathematics education. The literature on situated numeracy has had a significant impact in mathematics education since it challenges many of the dominant myths and assumptions that have underpinned significant components of mathematics. Two main areas of challenge are discussed.

First is the transferability of knowledge from one context to another. One of the more persuasive assumptions underpinning mathematics education is the notion that the mathematical thinking and reasoning embedded in school mathematics can be transferred to non-school contexts. This is a powerful belief and supports the status of mathematics within the formal school context and in the wider social arena. This position supports the notion that abstract concepts and operations learned with the school context can be transposed across other contexts. For example, if the student learns the basic operation of addition, eg 3+4=7, then this knowledge will be useful in contexts such as shopping (3 apples + 4 bananas = 7 fruit), fuel consumption (3 litres + 4 litres = 7 litres) or workplaces (3 metres of 4x2 hardwood + 4 metres of 4x2 hardwood = 7 metres of 4x2 hardwood). Lave (1988) and Walkerdine (1988) have challenged such positions by proposing that the everyday mathematical processes used in resolving what appear to be mathematical tasks are substantially different from those undertaken in the formal school context. Lave's (1998) study of shoppers has shown that processes of checking prices on cheeses does not align itself with the ratio problems taught in the school-context. Kanes (1996) study of motel check-in staff shows that the simple addition tasks involved in calculating accounts are influenced by the motel goods consumed. Similarly, the study of Brazilian street vendors (Carraher, 1988; Carraher, Carraher, & Schliemann, 1985) and candy sellers (Saxe, 1988) offer strength to such proposals. Through the research into everyday practices, the question as to whether knowledge does transfer from one context to another, and in particular whether the mathematical knowledge learned within the school context transfers to the everyday context, is challenged.

Second, is subsequent to this challenge. If indeed, mathematics is not transferable, then the status of mathematics is under threat. The status of academic knowledge over practical, everyday knowledge is threatened. School mathematics is high-status knowledge which maintains it power through assumptions that the knowledge is powerful because it is abstract, transcultural and transposable. The situated numeracy discourses raise challenges to these beliefs.

The move towards those situated learning approaches in which the learning and the social context are inextricably bound together so as to be mutually constitutive is useful in demonstrating the power of the context in determining the processes and knowledge used in task resolution. The skills used are often distinctly different from those learned in school maths, and moreover are more effective. Carraher et al's studies have shown the efficacy of situated numeracy and the inefficacy of school-maths. The activity which is undertaken within the workplace must be understood as a dialectical engagement: it is the self-organising and reciprocal arrangement between socially-constituted people and socially-constituted settings. Within such an approach, the goals commonly associated with school mathematics, such as abstraction and deductive reasoning, are not the same goals of the workplace, so that the forms of mathematics are different.

In studies of out-of-school contexts, research has valorised the practical knowledge and understandings commonly associated with "common sense". The Carraher et el studies indicate, for example, that the mathematising of practices often confuses students. In contrast, young students in their everyday contexts are able to conduct efficiently and effectively marketplace calculations but in the school context are unable to complete similar calculations. These studies suggest that the differentiation between the two practices and the legitimacy and status of school knowledge may be inappropriate. The implications of this research is the challenge to school mathematics to become more vocationally orientated and that since the context is profoundly interwoven with cognition, then learning should occur in contexts most like those to which the students will graduate. This has substantive implications for workplace reform in concert with school-maths reforms.

Applications and Situated Numeracy

In contrast to this research is the work founded on the notion that adopting an approach which uses everyday problems is restrictive to the development of substantive mathematical reasoning. Within this approach, "pure" mathematics is assumed that the more decontextualised the mathematics, the more likelihood there is of transfer. Sierpinska (1995, p.2) considers the implications of this work for school maths practices and argues for a position somewhere between the two extremes. She suggests that in order to be able to apply decontextualised mathematics to practical, everyday situations, students must be "taught the art of applications" (p. 3). She goes on to argue that it is not the distinction between the everyday and the school contexts which marks the difference but in the nature of the problems being given. In the everyday context, there is an authentic problem which requires the students to employ problem posing strategies such as those suggested by Silver (1994). In this problem posing context, authentic problems are generated by the students and in doing so, they are free to choose resolution strategies rather than being bounded by prescribed rules and procedures. Sierpinska (1995) argues that in order to complete the tasks within a context, particular rules of logic are called to task. In the case of school mathematics, very particularised practices are employed which have been learned over a period of time. In contrast, the methods used within the everyday context are substantially different, often not calling on even the simple mathematics of school. In order to complete the everyday tasks, Sierpinska proposes that the individual is "not so advanced in her mathematics. To get to this point she must first learn simple school arithmetic" (p. 5).

The differences between a workplace or non-school environment and a more traditional "school" environment can be observed in the forms and processes of mathematics which are employed by participants. Whereas formal school contexts adopt problem solving as an approach which offers considerable advantages over the more traditional "chalk and talk" teaching methods, the problem solving found in the workplace is substantially different from that of the school context. Workplace problem solving demands that learners apply situational mathematical knowledge and concepts to situations which are constrained by real-life problems. For example, Masingila (1993) suggests that her study of carpet laying requires not only the concept of area to be applied, but also real life constraints applied to the laying context which include the nap of the carpet and placement of seams. Masingila's (1993) studies of carpet layers lead her to suggest that the situated mathematics of this group of people is different from school mathematics in a number of key dimensions.

- The people are actively engaged in DOING mathematics. The constraints of their jobs pose REAL problems which must be solved.
- The apprenticeship model of learning provides a good method for teaching/learning the occupation.
- Textbooks could not provide the detailed constraints of the occupation which would hinder the problem-solving skills needed for the job.
- Learning-in-context creates a broader learning of a concept due to the active involvement in real problem solving tasks demanded by the job.

The issue which needs to be considered is whether school-based tasks actually represent the real life contexts which they are purported to do, or whether they are a veneer which serves to support the legitimacy of mathematics as a discipline which can be reappropriated and applied across many contexts. Masingila (1993) suggests that the complexity of problem solving demanded in certain occupations could not be represented in a text-book task and that often what it represented in such textbooks is a computational task masked in an everyday situation. The resolution of such tasks is dissimilar to the processes and skills that would be employed by a worker working with a real problem. Often the processes which are taught in formal contexts such as schools often prevent or discourage the development of links between workplace and school mathematics.

Situated learning has offered improved understandings of the cognitive processes used in the resolution of everyday tasks and challenged the myths of transferability of knowledge from the school context to the everyday, practical context. Sierpinska has raised the challenge to consider the authenticity of tasks - school and everyday - and the consequent mathematical skills and knowledge used in that context. While Lave's shoppers were very effective in solving the pricing of cheeses, the strategy may not have been as effective had a mathematical strategy been employed. Quite clearly there are situations which call for highly mathematical methods for resolution while others can be resolved through everyday methods which are not so heavily reliant on formal mathematical methods. To this end, the mathematics employed within a context influences and is influenced by the particular demands of the context. What is proposed in the remainder of this paper is the notion of mathematical saturation.

Mathematical Saturation

In the following sections, I develop the notion of mathematical saturation through the use of three distinct workplace environments. In each context, there are distinctly different forms of mathematics used and demanded. The choices of mathematics available to a participant are constrained by the background knowledge of the student thus lending support to Sierpinska's (1995) earlier comments about the need to be mathematically numerate to be able to operate effectively and mathematically.

In developing the notion of mathematical saturation, "mathematics" is seen to be the pure form of mathematics used in the formal school context. This is in line with Dowling's (1997) proposal of mathematical saturation in mathematics texts. Rather than propose a model in which there are, or are not, modes of mathematics operating (as in many of the approaches to situated numeracy); or that there are models in which pure mathematics is seen to be the most appropriate strategy resolution; or somewhere in the middle as proposed by Sierpinska, this model suggests that there is a continuum which spans from contexts where the amount of school mathematics is negligible through to contexts which are highly mathematical.

The following examples are taken from a research project in which employees were interviewed and then followed for a period of time by the researcher. The interviews consisted of ascertaining the employees conceptualisation of their work duties and what, if any, mathematics they used in their employ. A range of industries were involved in the project but only those involved in fast-foods, construction and nursing will be reported on in this paper.

Low Saturation- Crew member at MacDonalds

Contexts where there is little need for the application of formal mathematics to resolve the task can be seen to be low in saturation. In these contexts, the participant develops effective and efficient strategies for task resolution without embedding strong mathematical knowledge and skills. For example, the choices made by shoppers as to whether to purchase particular sized items based on constraints such as whether the containers will fit in the cupboards or whether the goods would be consumed within the stay-fresh period indicate low mathematical saturation. Clearly the problem solving strategies are highly contextual but they are low in mathematical content. Such situations would be solved in the formal mathematics contexts as ratio problems.

Crew members at MacDonalds are required to prepare and stock the counters. Most of the tasks are automated with very little autonomy for the employees. In spite of the need to measure time, quantities, money and estimation, most of these tasks are automated. For example, cooking pancakes requires the employee to push a handle and the contents are exuded on the griller in set amounts of set size. A timer sounds when the pancakes require turning. After turning and cooking the reverse side, the pancakes are then placed in the tray.

The employee comments aptly sum up the mathematical saturation in the workplace:

> Crew: Don't really use maths, just do it as MacDonalds says because it has to be the same all the time. You learn how they want it done and do it like that.

An examination of the workplace skills needed indicate some levels of mathematics but these are minimal. Handling money is highly electronic in that totals are calculated according the items pressed (rather than amount) and change calculated. Some counting skills are necessary to ensure the correct change is taken from the cash register. Similarly, in preparing the mixture for the pancakes, large packets of premix are required to be mixed with particular amounts of water. This water must be measured and then mixed into the prepared to mix to form the batter. More generic skills of problem solving are needed to ensure that there is a constant flow of food, but these skills are not necessarily the responsibility of the junior crew member.

When considering the context of the MacDonalds crew member, the situated numeracy is minimal and the degree of mathematics embedded in the tasks to be undertaken was very low. Most of the mathematics has become automated so that the need to quantify or measure was almost eliminated. Minimal counting was required – as in the water:pancake mix ratio – so that there was very little mathematical saturation.

Medium Saturation - Construction Work

In the second scenario, case studies of construction workers were undertaken. The study of pool builders provided evidence of highly contextual numeracy and problem solving with some use of formal school mathematics *albeit* relatively limited. It is therefore proposed that medium saturation was evident in the use of mathematics in this context, but it was not necessarily of the formal school mathematics. Mathematics was used, but it was modified for and by the context. The type/s of mathematics used varied and depended on the tasks being undertaken.

Various teams are used in the construction of pools which can be broadly conceived of as excavators whose task is to prepare the site for the pool including the excavation of the pool hole; box-and-framers whose task is to construct the free-standing frame for the pool; the concreters who are responsible for the formation of the pool; the finishers who apply the required finish (tiles, pebbles, sealer) as selected by the owner; and the minor roles of plumbers, electricians and supervisors of the whole project. In all phases of the building process, estimation skills feature strongly for all teams - estimating the amount of soil to be removed and the number of trucks needed to remove the dirt; estimating the amount of concrete needed and the application of that concrete. The estimation skills of most of the teams are well developed but are not heavily reliant on formal mathematics. Of particular importance in considering mathematical saturation are the skills of the box-and-framer who requires a mix of highly contextual numeracy skills alongside a number of formal mathematical skills.

The mathematical skills of the box-and-framer are developed and constrained by the context. The site most often is a large hole in the ground covered with a layer of screenings on the floor to allow for movement once the pool has been poured. For the box-and-framer to have access to measuring tools to mark out the positions of reinforcing bars in the box-and-frame stage would be constraining and clumsy. The framing of the pool is a three-dimensional project in which horizontal and vertical framing bars must be considered in conjunction with the flooring grid so that a vertical bar on the side of the pool will also form the flooring grid. All of these bars must run in parallel lines of equal spacing. In order to construct the frame for the pool, the box-and-framer needed to cut and bend the reinforcing bars at the appropriate places. Only the length of the bars for cutting were measured, but where bends were to be located were estimated visually. These bends needed to consider the 3 dimensions of the pool so that a bar needed to traverse the length of the pool, up the side of the pool and then splayed around the lip to provide appropriate spacing. Further more, in line with Lave's notions of experts and novices, the master box-and-framer is able to accurate estimate the placement of horizontal and vertical bars through the three-dimensional frame. For example, the 65 bars around the circumference of the pool must be positioned within 30 cm +/- 0.5 cms to comply with council regulations. There were only 5 bars which were marginally out of this range. Similarly, the placement of the lip of the pool must be 9 inches which was measured using hand spans.

The estimation skills used for measurement in this context were highly contextual and well developed. The accuracy was very high. Motivation to be accurate was high as the various phases in the construction process were sub-contracted out. Each subcontractor was responsible for his costs and hence profits. The constraints imposed by the local councils who inspected the pool at various stages of constructed also provided motivators since the pool construction had to conform with pool regulations if it were to be approved. Hence, the box-and-framer was motivated for accuracy not only to comply with building regulations, but also profits. In this context, the numeracy was highly situated and effective having developed out of the context. The box-and-framer had an extremely high degree of accuracy without relying on formal mathematics.

High Saturation - Nursing

As with other professions, nurses have developed numeracy skills which are situational. A number of aspects of nurses' work require various degrees of mathematics. Many of the traditional tasks requiring mathematical calculations have become redundant. For example, intravenous drips have been modified so that the tedious calculations for drip rates have been replaced with a type of sliding scale found on the IVAC pumps. This allows the nurse to enter the amount of saline drip to be consumed within the time nominated so eliminating calculations.

Where mathematics has become highly situational, nurses have developed idiosyncratic ways of documenting aspects of their work. This documentation is a unique component of the medical profession where not only has accountability become greater, but economic rationalism has meant that more patients need to be tended to in smaller amounts of time so quicker methods of patient assessment must be undertaken. Having patient information readily accessible in some format for doctors and changeovers means less patient-medico time. This documentation of patient-responding-to-treatment can be observed in wards where there is the need to monitor wound healing rates, such as ulcerated limbs. The nurse estimates the length, width and depth of the wound and then maps this onto a graph. Changes are noted over time, and the changes noted on formal documents. Similarly, in the obstetric wards, newborns are given an AGPAR score to indicate the overall health status of the child. This is an almost immediate score which consists of ratings which are on a 0, 1 or 2 scale which are then summated to give an overall score out of ten. This is then calculated again after a short period of time to ensure that the child is maintaining or improving in health status. This score is often an immediate response, rather than the mathematical task of school mathematics. There is minimal mathematics embedded in the tasks, but rather "professional judgement" is seen to be more important. There is no formal measurement of wounds or infant colour, but rather the judgement of the nurse is the formal documentation. Nurses were very confident in this aspect of their work.

One aspect of nursing containing a high level of mathematical saturation is problematic for many nurses. Unlike other aspects of the profession where situated numeracy skills can be used and developed, the ratio problems of drug consumption are highly mathematical. For example, in post-operative care, the patient may be required to consume Heparin which is a drug used as an anticoagulant in order to prevent DVT (deep vein thrombosis). The doctor may prescribe a dosage of 3000 units per ml but it is stocked at a strength of 5000 units per ml. In order to give the patient the correct dosage, precise calculation needs to be undertaken. Over-dosage may result in death so motivation should be considered as extremely high. As in other contexts, motivation provides a strong context for ensuring "correct" resolution of tasks yet in this context, the mathematics subsumes the meaningfulness of the task and hence is often undertaken as a meaningless operation as noted in the comment made be a nurse:

> Nurse: To work out what I have to give, I get "strength required' over "strength in stock" multiplied by volume. If I have to work out a Hep [Heparin] dose, I would have 3000 over 5000 multiplied by 1 ml. That gives me 0.6 ml.
>
> R: How did you work that out?
>
> Nurse: It is one we do all the time, so I just know it. I get a bit confused with the way to do it if I get a new one. I have to think about which number is on the top and which is on the bottom.

The lack of confidence in understanding the actual processes involved in working through the problem indicates the nurse's conceptual understanding of what is actual being undertaken. The high degree of reliance on formulae is reminiscent of school mathematics. Unlike the box-and-framer where motivation is closely linked to profit and loss, nurse's calculations are often life and death situations so it would be expected that there would be higher motivation in this context and hence greater motivation for task resolution. Yet as nurse educators and hospital administrators are acutely aware, this aspect of nurse professional development is critical. yet the degree of mathematical saturation hinders nurses' capacity to effectively undertake the task.

Conclusion

The types of mathematics used and developed within a context appears to have a significant bearing on the confidence of the participants and their subsequent efficacy. Where there is high degree of mathematical saturation, there is less chance of the participants feeling confident and empowered with that situation. Indeed, as nurses have reported, they are not confident in using the mathematics for calculating ratios for drug use in spite of its high degree of motivation - being a life and death situation. In contrast, where the level of mathematical saturation is substantially lower and the mathematics has been developed within a context such as the box-and-framer's placement of bars and the nurses' AGPAR scores, there is a great sense of confidence in the results.

The notion of mathematical saturation needs to be developed further as it has merit in the context of situated numeracy. From the examples used in this paper, there are varying degrees of mathematics embedded in the workplaces observed. In each setting, task motivation varies along with the degree of mathematics being employed. What becomes clearer is what is considered to be "mathematics" and how this then becomes conceptualised as situated numeracy is of great importance in this area of adult and workplace learning.

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ABSTRACTS OF SHORT COMMUNICATIONS

Realistic contexts as critical sites for problem solving

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The paper discusses one of three case studies to investigate students' conceptualisations of quadratics by solving quadratic contextual problems. The theoretical framework which guided the study was constructivism. The methodology was teaching interviews between a researcher and student solving problems. A multi-representational software FUNCTION PROBE was available as a tool to aid students in their problem solving. The results showed that using realistic situations as problem contexts invited a multiplicity of interpretations and methods for defining quadratic functional relationships. By reasoning empirically from the context of the problem, the student conceptualised quadratic relationships iteratively and in terms of summation in contrast to the most common view of quadratics as a product of two linear variations. Further, requiring students to verify and justify their strategies by cross-referencing between multiple representations of functional relationships and problem context led them to construct viable schemes to characterise quadratics in terms of rate of change, dimensionality, and symmetry. This paper describes the case study of a student who generated difference equations as a means of predicting maximum values of quadratics mainly as a result of her struggles to reconcile her conjectures and errors in her numerical values. Her subsequent resolution of errors motivated her to pursue an alternate approach to the problem which led her to derive a more elegant form of her quadratic function in contrast to her initial summative expression.

Teachers' views of collaborative learning in secondary classrooms

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A survey was conducted of teachers who make use of collaborative learning approaches in teaching secondary mathematics. Collaborative learning is taken to mean forms of classroom organisation in which students work together in small groups, on shared tasks and with common goals. No particular organisational or assessment structure is implied. The questionnaire aimed to discover how teachers implement collaborative learning in mathematics, some of their reasons for using it (or not using it) at different levels, and their impressions of how students respond to learning in this way. This paper reports the results of the survey.

Grounding knowledge and skills with understanding and meaning in mathematics education: A preliminary report

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This short communication offers a preliminary report on one component of a National Science Foundation Reform Initiative for Undergraduate Education in Science, Mathematics, Engineering, and Technology at the University of California, Irvine. This component involves the development of a new course in mathematics education for prospective teachers, with the intent of grounding mathematical knowledge and skills with understanding and meaning. The report focuses upon the theories and methods used in researching the design, implementation, and evaluation of this course.

Record keeping by teachers of mathematics: A search for meaningful and manageable purposes and practices in assessment

Barbara Clarke **Doug Clarke** **Donna Gronn**
Monash University **Australian Catholic University**

Multiple modes of assessment provide rich information about student learning and "capturing" a range of performances. However, the challenges of meaningful and manageable record keeping are considerable, and the research and professional literatures provide little support. This project is using questionnaires, semi-structured interviews, collections of relevant documents, and the advice of an expert panel of teachers to document a range of promising models of record keeping currently in use in schools, with a view to identifying and sharing exemplary practice.

Evaluation of the mathematics node of NSW HSC On-Line
(A report of the first six months)

Bob Dengate
School of Teacher Education, Charles Sturt University, Bathurst

The NSW On-Line Project was initiated by Charles Sturt University (CSU) in 1996 as a resource for senior school students, teachers and relevant community and professional bodies. It was developed as a joint venture with the NSW Department of Education and Training (DET); the partners were successful in winning an Access Australia CMC Research Seeding Grant to evaluate the development of the website and provide recommendations for further development. Central themes emergent from the evaluation included general satisfaction with the site's appearance and structure, excitement and enthusiasm about its potential as an educational resource and the need for future promotion of the site. An outcome of the research was the need to further explore appropriate online pedagogies and to determine models to train and develop teachers in the effective use of the medium compared with traditional techniques. Research recommendations regarding strategic planning, technical development and implications for best teaching practice need now to be contextualised so that, for example, implications for Mathematics Education can emerge. The author, who coordinates the Mathematics Node of the site, seeks ongoing feedback and advice regarding publication, structure, pedagogical design and strategic research planning for the site's Mathematics Node, in particular, and for the Project generally.

A mathematical inquest: Explaining a negative outcome of instruction

Carmel Diezmann
Queensland University of Technology

After instruction in the strategy draw a diagram, there was a decline in the number of Year 5 students who were successful on a deductive reasoning task. Although this result suggested that the intervention adversely affected students' performances, an investigation revealed that most students benefited from the intervention by developing some understanding of diagram use in problem solving. The anomalous result is explained by spontaneous change in students' strategies and their differing proficiency in the use of a diagram.

Challenges facing tertiary mathematics: Lecturer's views

Helen Forgasz
La Trobe University

Downsizing in the tertiary sector, the apparent lack of appeal of careers in mathematics and recent patterns of declining enrolments in majors and postgraduate degrees have placed tertiary mathematics departments under increasing pressure. In this paper, the views of six mathematics lecturers who spoke frankly about the roles of their departments and the challenges ahead are presented. Concerns for the discipline of mathematics and for students were evident. Despite a climate in which budgetary constraints dominate over teaching and learning considerations, there was some optimism for the future.

An investigation into the use of active reading skills in individualised mathematics diagnostics and their impact on mathematics education

Freda Goodall
Victoria University of Wellington

In testing the diagnostics with students from seven to fourteen years it was found that the majority exhibited difficulties in three aspects of reading: reading accurately, recall, and application of content. Questions beginning with : who, what, when, where, why, how, were constructed for use in the diagnostic. The experiment indicated that the mathematics results of the students in the experimental classes improved when compared with those not receiving assistance in reading.

The importance of validity in test construction

Freda Goodall
Victoria University of Wellington

This paper examines the validity of the diagnostics (tests) for the purposes they are designed for, which are to allow the teacher to understand the levels of learning a child or student has reached in mathematics education, at a particular point in time. To prove the validity of the diagnostics, the work of Crooks et al. (1995) will be used to illustrate that the tests perform the functions for which they are designed.

What are approaches to learning statistics?

Sue Gordon
Mathematics Learning Centre, The University of Sydney

In this presentation I discuss my development of a questionnaire — the Approaches to Learning Statistics Questionnaire. I argue that while surface approaches are often equated with rote learning, this is not what characterises these approaches. Instead, it is the learner's intention that is important. I discuss facets of deep approaches to learning statistics emerging from my questionnaire. My data support a systemic view of teaching and learning in context. They raise issues concerning the support of statistics education at university.

Learning about probability in a constructivist classroom

Chris Linsell
Dunedin College of Education

When students engage in rich mathematical activities our intention is that they should construct their own mathematical knowledge consistent with the achievement objectives of the lesson. Four Year 9 students were videorecorded during lessons on probability. The videorecordings were then used to conduct stimulated recall interviews with each student. Preliminary analysis of the data shows a wide range of achievement, with even the most able students finding some concepts difficult to grasp.

Teaching area in Year 1 and Year 2 classrooms

Diane McPhail
University of Western Sydney, Macarthur

This paper will give a progress report on the first part of a project which aims to improve student learning outcomes in the Area substrand of primary school mathematics, through enhanced teacher awareness and an understanding of how students may be assisted to develop Area concepts. This initial project will trial a sequence of four lessons in each of four classrooms, focusing on recognising and calculating repeated units (squares and triangles) within a grid pattern. Evidence of student learning will be documented through observations of activities, collection of work samples, videotaped interviews with students and audio taped interviews with class teachers.

Changing practice towards improving learning in mathematics: A teacher's story

Lyn Nothdurft

This paper discusses an action research study focused on approaches to improve learning with a class of Queensland Year 12 Mathematics B students. There is evidence that, for many of these students, the experience of school mathematics involves following rules without understanding and/or enjoyment. In this study the students are being encouraged to actively seek deep understanding and take control of their own learning. The teaching strategies emphasise making sense of the mathematics, collaborative group work with a problem-solving approach, and student reflection on learning. Current indications are that the students are responding well to this approach.

Enriching learning through investigative and collaborative teaching in a senior secondary classroom

Bruce Nulty
Centralian College, Alice Springs

"Interpretive" research methodology was used for a small scale ethnographic case study of investigative and collaborative teaching strategies for one topic (coordinate geometry) in one Year 11 mathematics class. This was a qualitative study using anecdotal data. The findings supported the literature, which indicated that qualitative differences in student outcomes were likely to be observed. Furthermore there was likely to be more of a focus on student understanding and appropriate use of graphics calculators and computers.

A framework for assessing and fostering students' statistical thinking

Ian Putt
James Cook University

Bob Perry
University of Western Sydney - Macarthur

Graham Jones, Carol Thornton, and Cynthia Langrall
Illinois State University

The short communication will outline an initial framework which has been developed from our observations of Year 1 to 5 students' statistical thinking over a one-year period, and previous research (e.g. Curcio, 1987; Friel, Bright, & Curcio, 1997; Pereira-Mendoza, 1995; Mokros & Russell, 1995; Wainer, 1992). Analysis of preliminary data obtained via a structured interview protocol has allowed us to refine the set of descriptors of the four key constructs of the framework. These key constructs, reading data displays, organising data, displaying data, and analysing data, have been adapted from the four data handling concepts identified by Shaughnessy, Garfield and Greer (1996).

In the presentation and the accompanying paper we will outline the descriptors and give some examples of the protocols obtained from the structured interviews with students at the various grade levels. Exploratory analysis of the data suggests the existence of four levels of statistical thinking across the four constructs: idiosyncratic, transitional, quantitative, and analytical.

Fundamental concepts in algebraic thinking at the junior high school algebra level

R. E. Roberts
Nambour State High School

The substance of this paper is a subset of the substance of a larger study of the cognitive and affective characteristics of high and low achievers in Year 10 algebra. The major study is concerned with curriculum development and addresses the key questions, "What is it that enables some students to succeed in school algebra while others fail?", and, "Are there key concepts which underlie algebra, and algebraic thinking, and which teachers are missing the opportunity to emphasise; and which are possessed by successful students of algebra, and not by the others?".

An analysis of school algebra curricula from Australia, the UK, and the United States appears to indicate that there are such concepts. Written tests of these concepts were developed, and taken by a sample of high and low achieving Year 10, state school and private school, students from a semi-rural community. The results, and some possible implications for curriculum development, of these tests will be discussed, and the broader context of the study - which includes an affective data questionnaire, and a algebraic problem solving interview - will be discussed.

Multimedia for mathematics teacher education

Kaye Stacey and Dianne Chambers
University of Melbourne

We will report on two initiatives using multimedia to enhance the experience of face-to-face teaching for pre-service education students, funded by the University of Melbourne and being conducted by a team from our department. Multimedia presents new opportunities for teacher education, including bringing "virtual children" into the lecture and tutorial room. Research associated with the use of multimedia is developing around several themes, including the degree to which the case presented should be "cleaned" before use and the useability of the resources, which have enhanced discussions in workshops.

For several years mathematics has endured substantial horrible ill treatment
- We need to act now!

Philip Swedosh
The University of Melbourne

Mathematics is under attack. Far less time is now allocated to mathematics in schools. About 50% of Years 7-8 and 20% of Years 9-10 mathematics classes are taught by individuals who have never studied a tertiary mathematics subject. Fewer students are studying mathematics at upper secondary and tertiary levels. Many students receive advice which turns them away from mathematics and affects their future course and career possibilities. Each of these issues has serious consequences and action should be taken now.

Some effects of self-assessment using the Self Test package on student learning

Janet Taylor
University of southern Queensland

In this paper we present the results of trialing a new flexible self assessment package called Self Test. This package has been designed primarily for distance students to assess their readiness or mastery of particular topics. It differs from other packages in that it allows a student to mark a series of questions as if they were an expert tutor giving them the flexibility to complete a problem in up to six different ways and to be credited when only part of the problem is accurate. Results of the evaluation of the package with groups of engineering students studying mathematics are presented briefly and the implications of how students utilise this type of package to learn mathematics are discussed.

An outline of forming a phenomenology of writing-to-learn mathematics

Andrew Waywood
Australian Catholic University

In this paper an overview is offered of a study of writing-to-learn mathematics which was undertaken as a doctoral thesis. The study illustrates an application of a phenomenological method to a corpus of writing-to-learn journals produced by seventeen Year ten students at a catholic secondary girls' school. The outcomes of the study to be emphasised in this paper are a methodological refinement of the phenomenological method, and, a repositioning of the theoretical understanding of teaching and learning away from the cognitive sciences and towards the human sciences.

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